

What is a number, and what should it be?

Richard Dedekind

Finding an Analogy

Aaron Stockdill

Mateja Jamnik, Daniel Raggi

Peter Cheng, Grecia Garcia Garcia, Holly Sutherland

Gaussian Sum

Gaussian Sum

$$\sum_{i=1}^n i = \frac{n(n+1)}{2}$$

Gaussian Sum

$$\sum_{i=1}^n i =$$

Gaussian Sum

$$\sum_{i=1}^n i = 1 + 2 + \cdots + n$$

Gaussian Sum

$$\begin{aligned}\sum_{i=1}^n i &= 1 + 2 + \cdots + n \\ &= n + \cdots + 2 + 1\end{aligned}$$

Gaussian Sum

$$\begin{aligned} \sum_{i=1}^n i &= 1 + 2 + \cdots + n \\ &= n + \cdots + 2 + 1 \end{aligned}$$

Gaussian Sum

$$\begin{aligned}\sum_{i=1}^n i &= 1 + 2 + \cdots + n \\ &= n + \cdots + 2 + 1 \\ &= \frac{1}{2} \times n \times (n + 1)\end{aligned}$$

Gaussian Sum

$$\begin{aligned}\sum_{i=1}^n i &= 1 + 2 + \cdots + n \\ &= n + \cdots + 2 + 1 \\ &= \frac{n(n+1)}{2}\end{aligned}$$

Gaussian Sum

$$\sum_{i=1}^n i = \frac{n(n+1)}{2}$$

Gaussian Sum

$$\sum_{i=1}^n i =$$

Gaussian Sum

$$\sum_{i=1}^n i = \frac{n(n+1)}{2}$$

Gaussian Sum

$$\sum_{i=1}^n i = \frac{n(n+1)}{2}$$

Gaussian Sum

$$\sum_{i=1}^n i = \begin{array}{ccc} & \circ & \\ & \circ \circ & \\ \circ & \circ \circ & \circ \end{array}$$

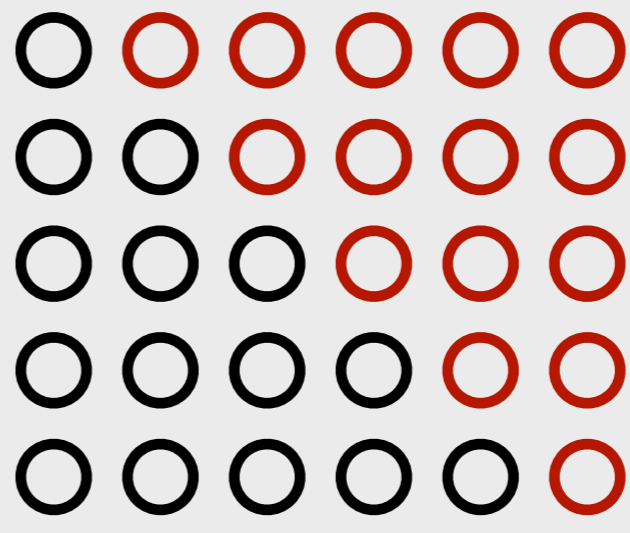
Gaussian Sum

$$\sum_{i=1}^n i = \begin{array}{cccc} & & & \circ \\ & & & \circ \quad \circ \\ & & & \circ \quad \circ \quad \circ \\ & & & \circ \quad \circ \quad \circ \quad \circ \end{array}$$

Gaussian Sum

$$\sum_{i=1}^n i = \begin{matrix} & & & & \circ \\ & & & & \circ & \circ \\ & & & & \circ & \circ & \circ \\ & & & & \circ & \circ & \circ & \circ \\ & & & & \circ & \circ & \circ & \circ & \circ \end{matrix}$$

Gaussian Sum

$$\sum_{i=1}^n i =$$


A 5x6 grid of circles representing the sum of integers from 1 to 6. The circles are arranged in 5 rows and 6 columns. The first row has 6 circles, the second has 5, the third has 4, the fourth has 3, and the fifth has 2. The circles in the second, third, and fourth rows are colored red, while the circles in the first and fifth rows are black.

Gaussian Sum

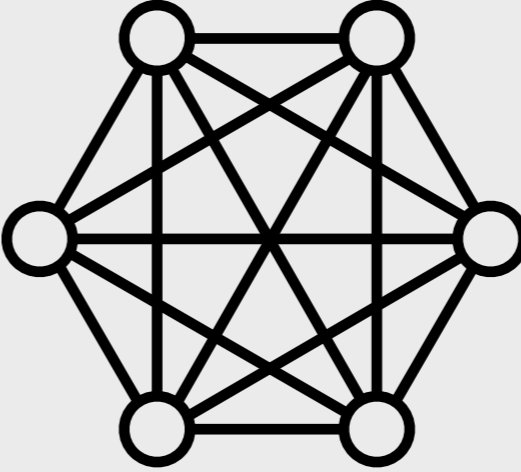
$$\sum_{i=1}^n i = \begin{array}{cccccc} \circ & \circ & \circ & \circ & \circ & \circ \\ \circ & \circ & \circ & \circ & \circ & \circ \\ \circ & \circ & \circ & \circ & \circ & \circ \\ \circ & \circ & \circ & \circ & \circ & \circ \\ \circ & \circ & \circ & \circ & \circ & \circ \end{array} \left. \vphantom{\sum} \right\} n$$

Gaussian Sum

$$\sum_{i=1}^n i = \underbrace{\begin{array}{cccccc} \circ & \circ & \circ & \circ & \circ & \circ \\ \circ & \circ & \circ & \circ & \circ & \circ \\ \circ & \circ & \circ & \circ & \circ & \circ \\ \circ & \circ & \circ & \circ & \circ & \circ \\ \circ & \circ & \circ & \circ & \circ & \circ \end{array}}_{n+1} \left. \vphantom{\begin{array}{cccccc} \circ & \circ & \circ & \circ & \circ & \circ \\ \circ & \circ & \circ & \circ & \circ & \circ \\ \circ & \circ & \circ & \circ & \circ & \circ \\ \circ & \circ & \circ & \circ & \circ & \circ \\ \circ & \circ & \circ & \circ & \circ & \circ \end{array}} \right\} n$$

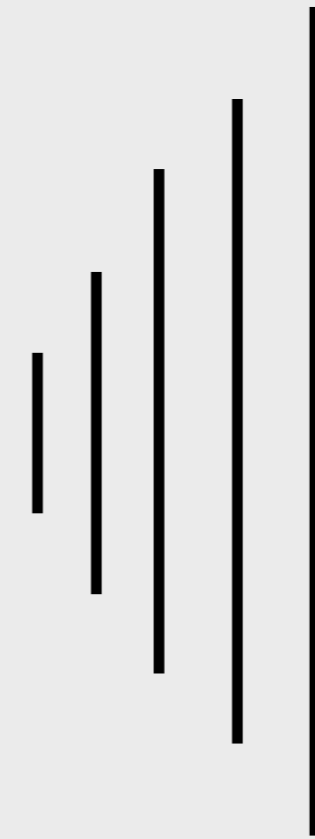
The diagram illustrates the Gaussian sum formula. On the left, the sum $\sum_{i=1}^n i$ is shown. On the right, a grid of circles represents the sum. The grid has 5 rows and 6 columns. The top row has 6 circles, the second row has 5, the third has 4, the fourth has 3, and the fifth has 2. The circles in the second, third, and fourth rows are colored red, while the others are black. A bracket on the right side of the grid indicates that there are n rows. A bracket below the grid indicates that there are $n+1$ columns.

Gaussian Sum

$$\sum_{i=1}^n i =$$


The diagram shows a complete graph with 5 nodes arranged in a regular pentagon. Every node is connected to every other node by a straight line edge, resulting in a total of 10 edges. This graph is denoted as K_5 .

Gaussian Sum

$$\sum_{i=1}^n i = \text{|||}$$
The diagram consists of four vertical black bars of increasing height, arranged from left to right. The first bar is the shortest, the second is taller, the third is taller still, and the fourth is the tallest. This visualizes the sum of the first four positive integers: 1 + 2 + 3 + 4.

Why?

Why?

People think differently

Why?

People **think** differently

Same content, different **cognitive features**

Why?

People **think** differently

Same^{*} content, different **cognitive features**

Why?

People **think** differently

Same content, different **cognitive features**

Switch to a more **accessible** representation

Why?

People **think** differently

Same content, different **cognitive features**

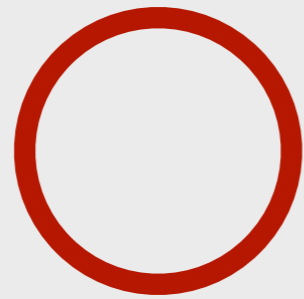
Switch to a more **accessible** representation

Expose **conceptual links**

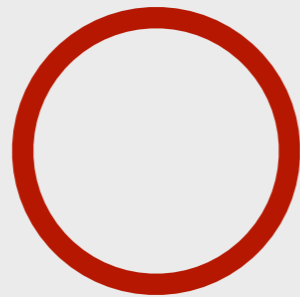
Structure Mapping

Structure Maps

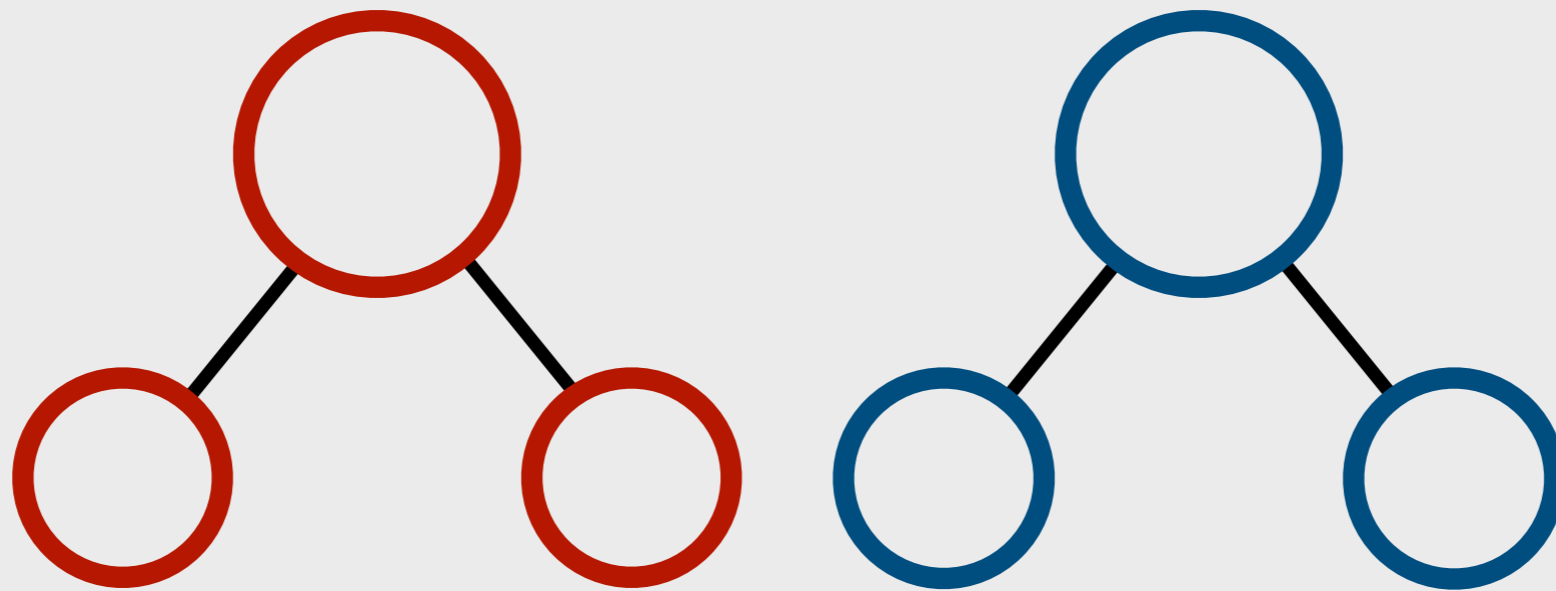
Structure Maps



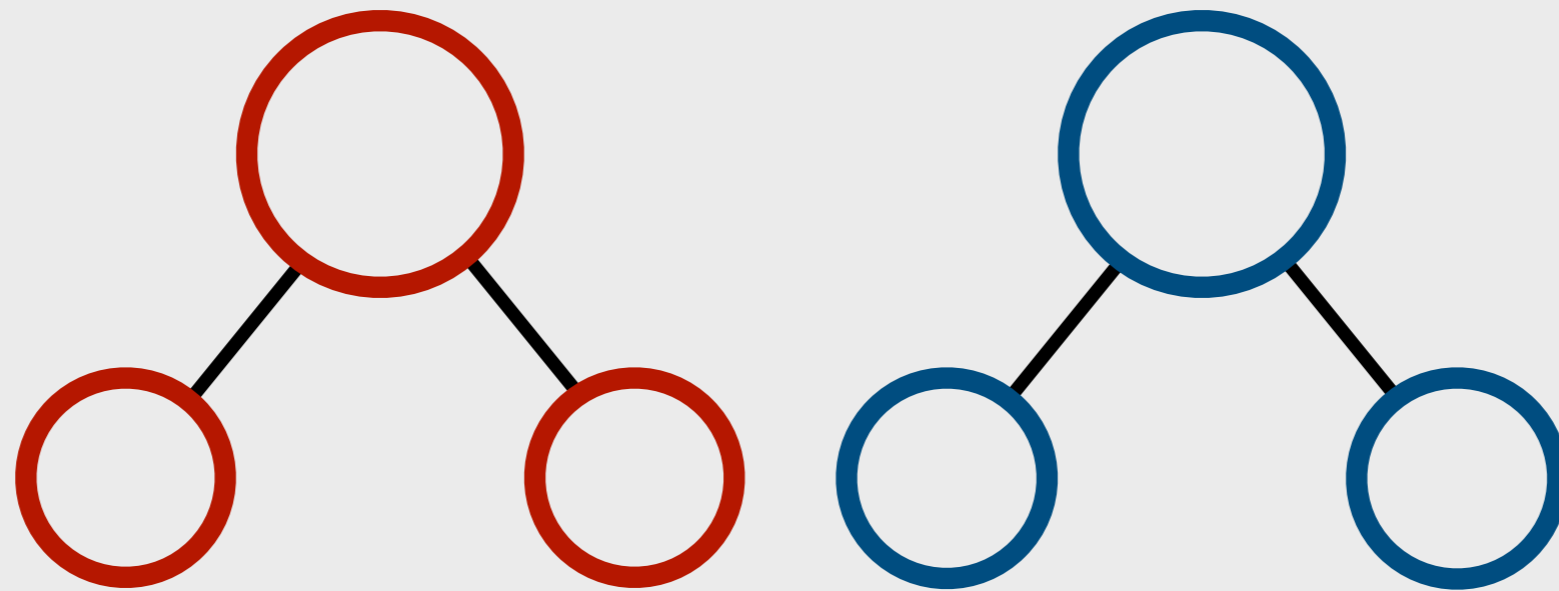
Structure Maps



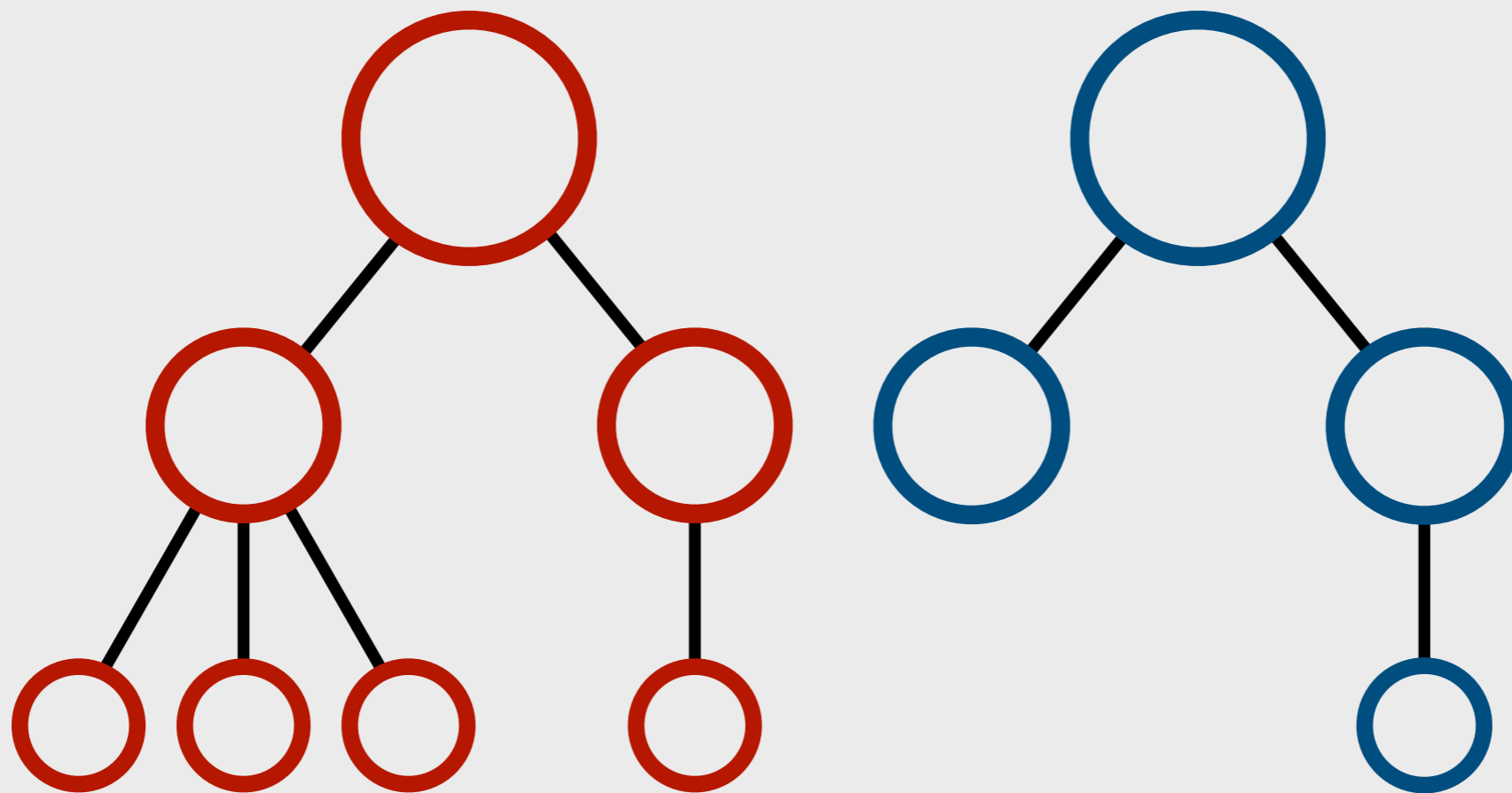
Structure Maps



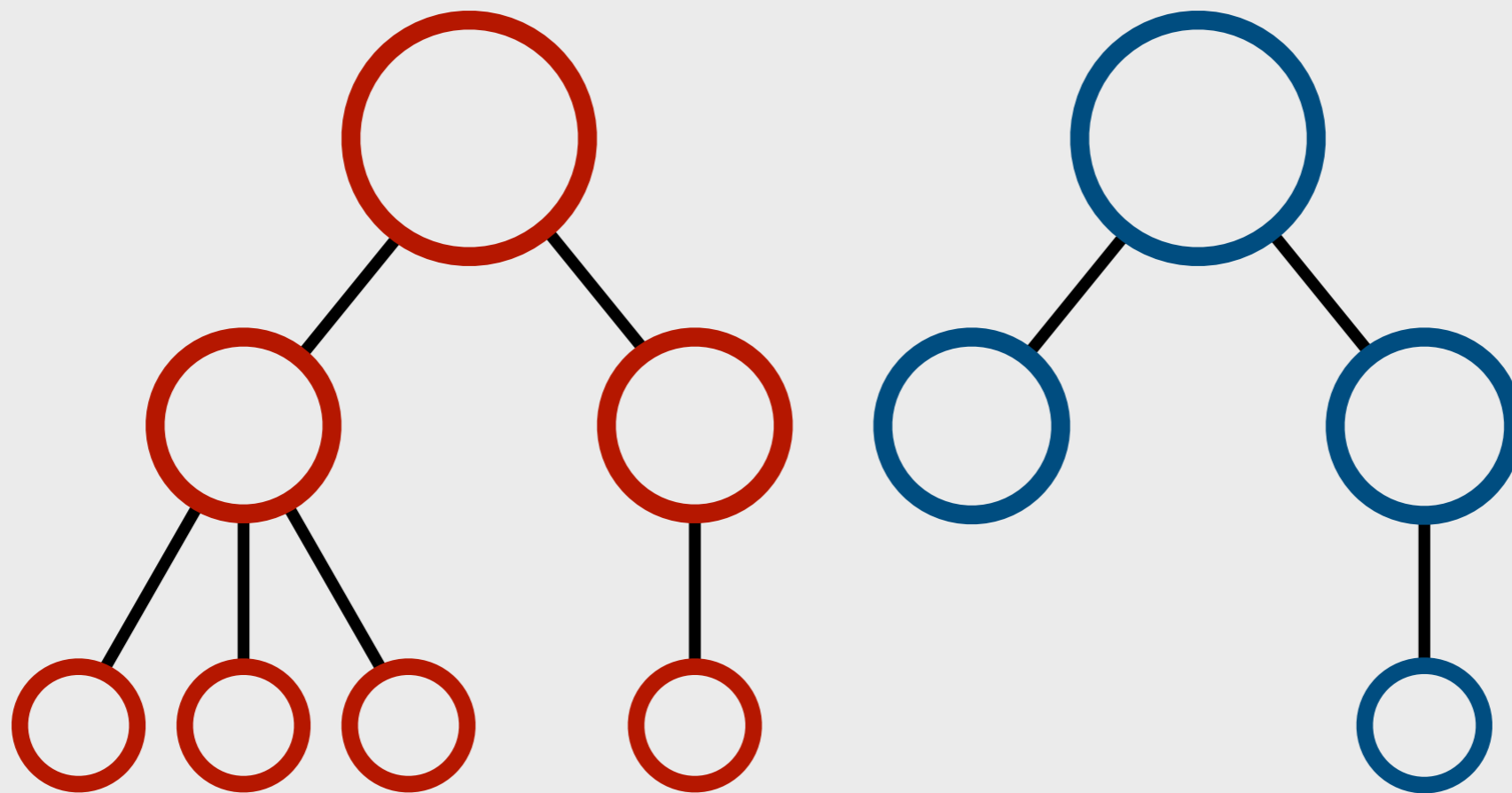
Structure Maps



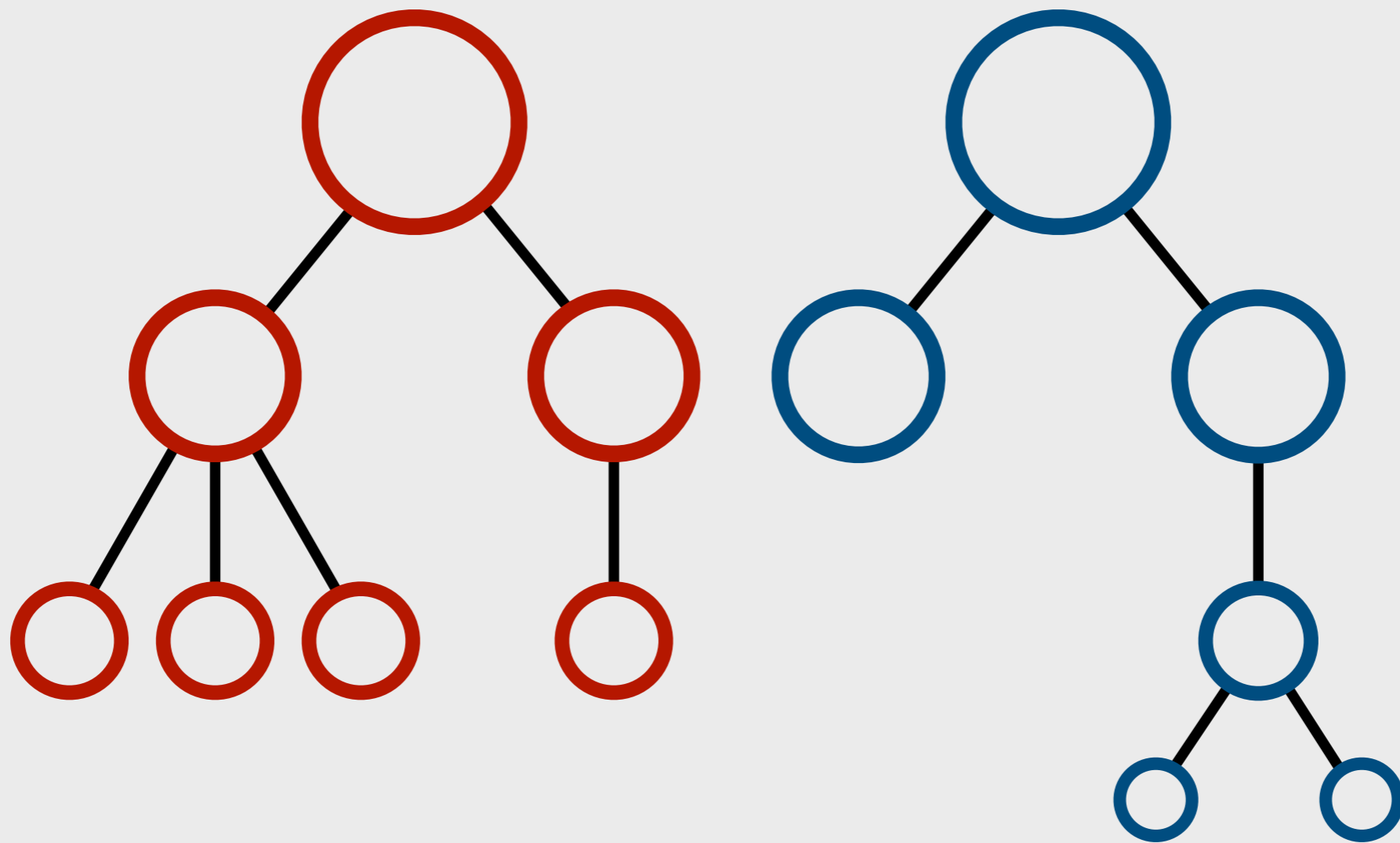
Structure Maps



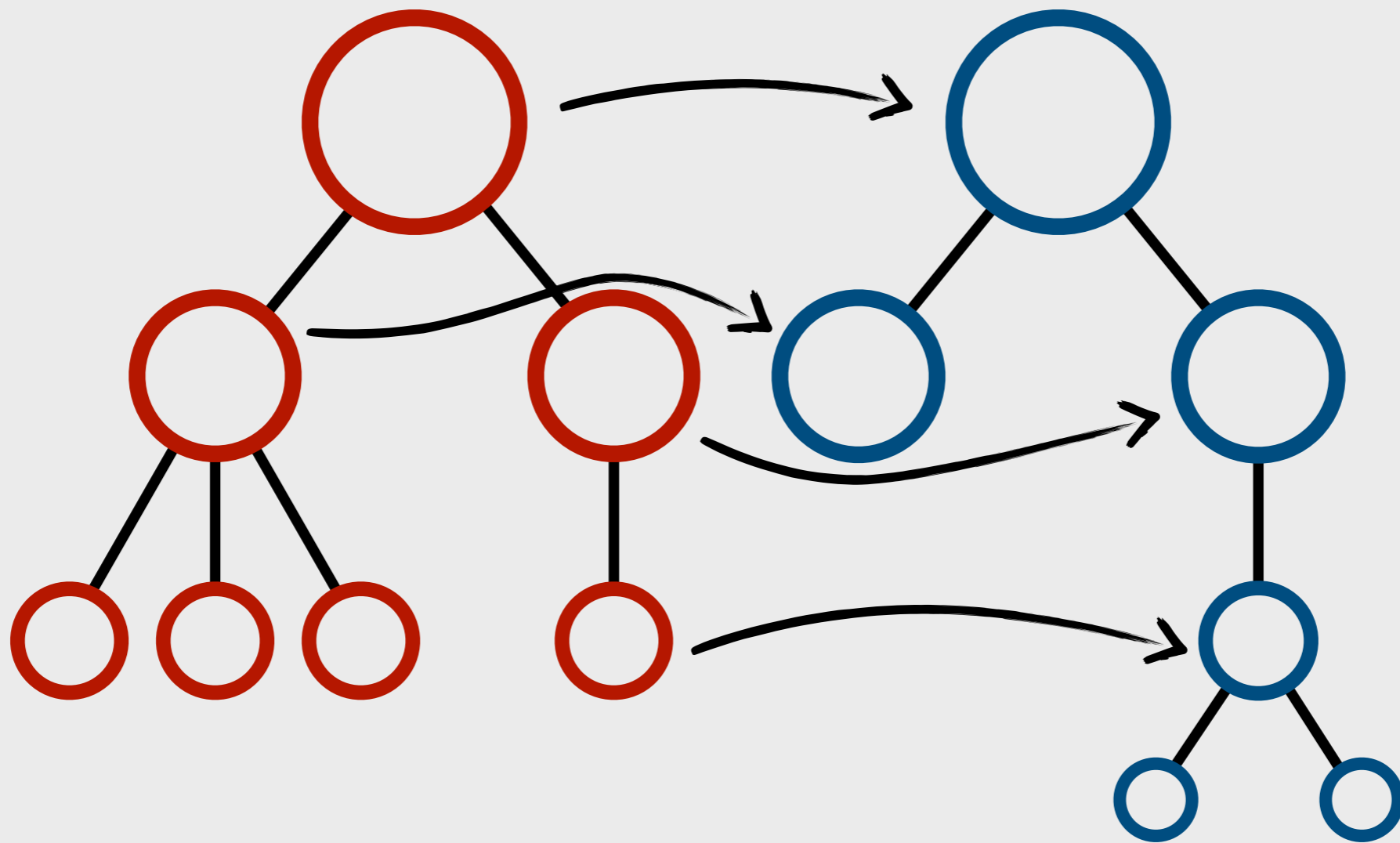
Structure Maps



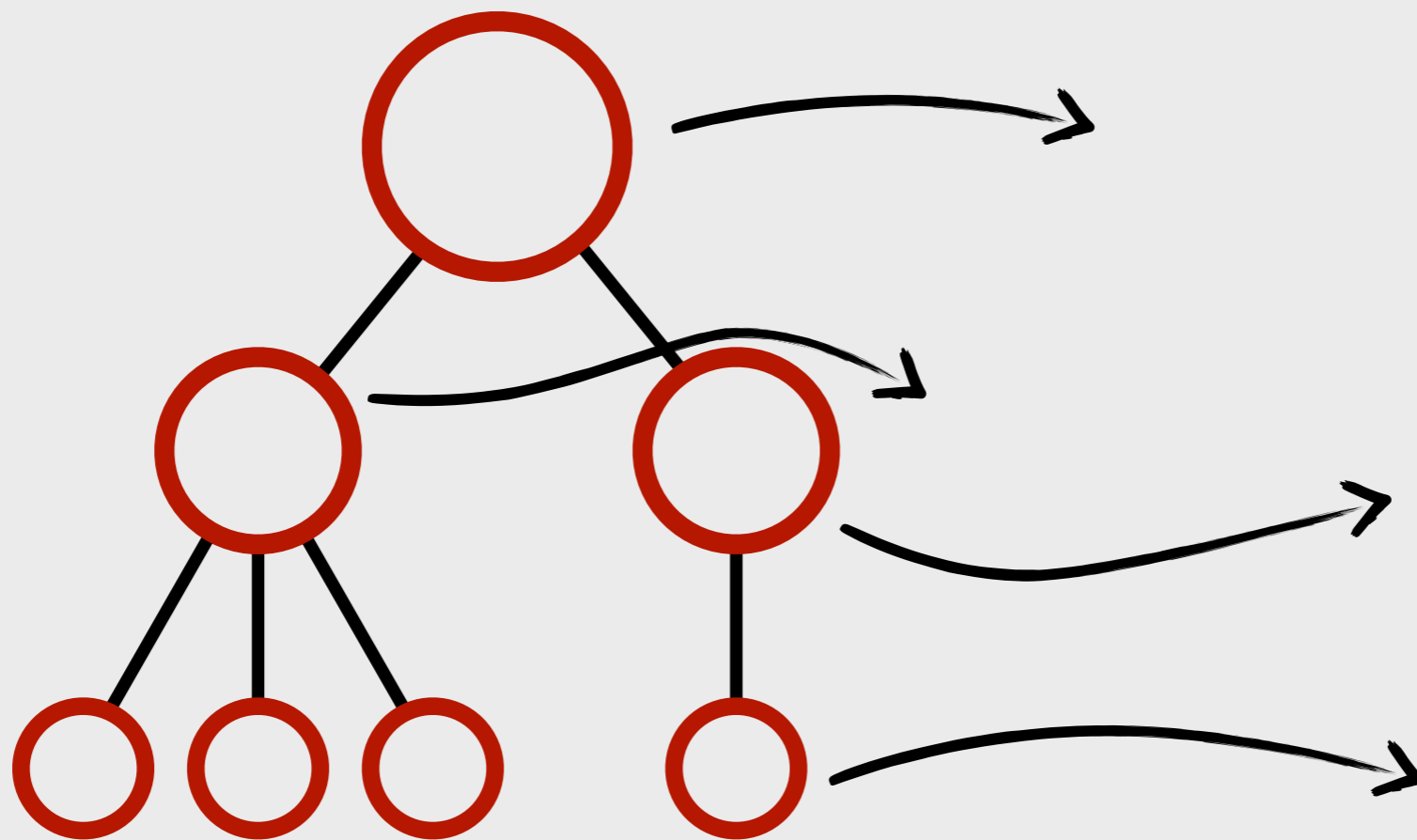
Structure Maps



Structure Maps



Structure Maps



Rules of the Game

Rules

Given statement in **first** representation

Rules

Given statement in **first** representation

Set of **alternative** representations

Rules

Given statement in **first** representation

Set of **alternative** representations

No available **translations**

Rules

Given statement in **first** representation

Set of **alternative** representations

No available **translations**

Small/no applicable datasets

Describing a Representation

Representations

Cohesive set of tokens, types, tactics, patterns, and laws

Representations

Cohesive set of **tokens, types, tactics, patterns, and laws**

Intuitive **boundaries** (usually)

Representations

Cohesive set of **tokens, types, tactics, patterns, and laws**

Intuitive **boundaries** (usually)

Can be **combined** in complex ways

Representations

Cohesive set of **tokens**, **types**, **tactics**, **patterns**, and **laws**

Intuitive **boundaries** (usually)

Can be **combined** in complex ways

Need to **describe** them!

```
1  let Algebra = representation
2
3  import tokens from real_numerals;
4  import tokens from latin_alphabet;
5
6  mode sentential;
7  rigorous true;
8
9  types integer, real, formula, proof;
10
11
```

13

14

15 **tokens** |, =, >, < **where**

16 type = integer → integer → bool;

17 **tokens** +, -, *, ÷, ^ **where**

18 type = integer * integer → integer;

19 **token** Σ **where**

20 type = 'a set → ('a → integer)

21 → integer;

22

23

24

25

26
27
28
29
30
31
32
33
34
35
36
37
38

```
pattern binaryOperation where
  holes = {integer: 3,
           integer * integer
           → integer: 1},
  tokens = [=];

laws +associative, +commutative,
     *associative, *commutative, ...;
```

39

40

41 **tactic** rewrite **where** laws = 1,

42 **patterns** = 1;

43 **tactic** calc **where** laws = 0,

44 **patterns** = 1;

45 **tactic** induction **where** laws = 2,

46 **patterns** = 1;

47

48 **end**;

49

Correspondences

Correspondences

Links **between** representations

Correspondences

Links **between** representations

What **fill the same role**?

Correspondences

Links **between** representations

What **fill the same role**?

Problem-independent

Correspondences

$\langle q, r, s \rangle$

Correspondences

q

r

s

Correspondences

q First representation properties

r

s

Correspondences

q First representation properties

r Second representation properties

s

Correspondences

q First representation properties

r Second representation properties

s Relationship strength

Correspondences

$\langle q, r, s \rangle$

Correspondences

$\langle q, r, s \rangle$

type number

Correspondences

$\langle q, r, s \rangle$

type number

type dot arrangement

Correspondences

$\langle q, r, s \rangle$

type number

type dot arrangement

0.9

Property Formulae

Formulae

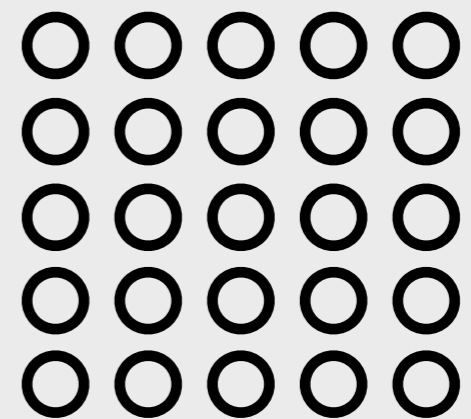
+

Formulae

+

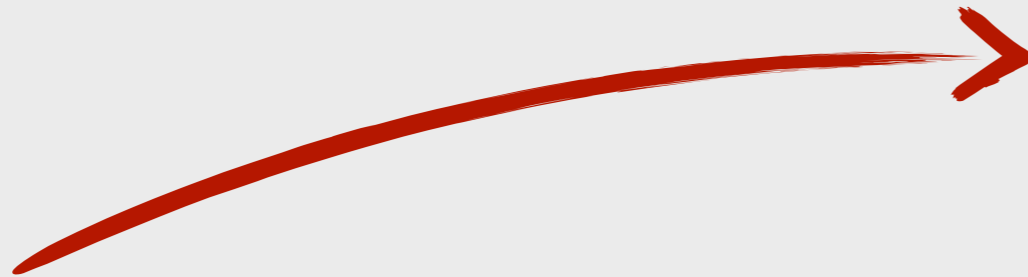


stack vertically

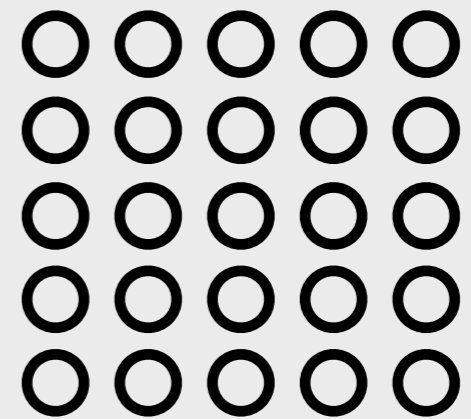


Formulae

+

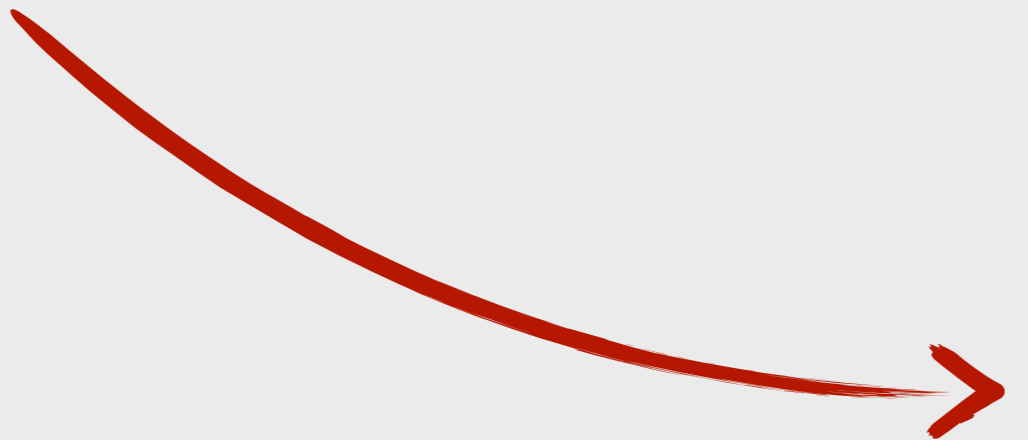


stack vertically

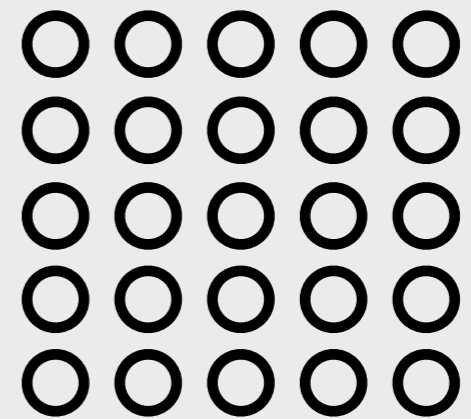


Formulae

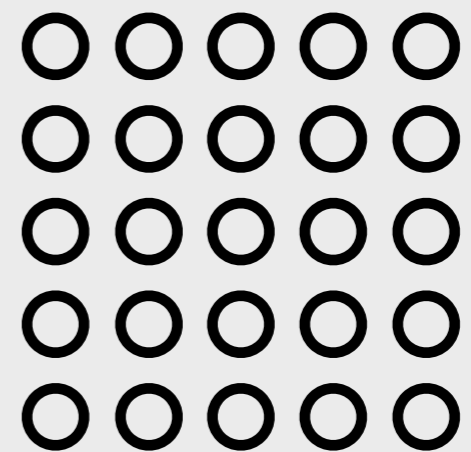
+



stack vertically



stack horizontally



Formulae

Alternative related properties

Formulae

Alternative related properties

Requires *several* properties together

Formulae

Alternative related properties

Requires *several* properties together

Properties should be *absent*

Formulae

Alternative related properties **OR**

Requires *several* properties together

Properties should be *absent*

Formulae

Alternative related properties **OR**

Requires **several** properties together **AND**

Properties should be **absent**

Formulae

Alternative related properties **OR**

Requires **several** properties together **AND**

Properties should be **absent** **NOT**

Formulae

Formulae

< token + ,

Formulae

< token + ,

tactic stack-horizontal
OR tactic stack-vertical ,

Formulae

< token + ,

tactic stack-horizontal
OR tactic stack-vertical ,

0.9 >

Strength

Strength

< token + ,

tactic stack-horizontal
OR tactic stack-vertical ,

0.9 >

Strength

< token + ,

tactic stack-horizontal
OR tactic stack-vertical ,

0.9 >

Strength

Measure of **suitability**

Strength

Measure of *suitability*

Perfect is 1, *meaningless* is 0

Strength

Measure of **suitability**

Perfect is 1, **meaningless** is 0

Any **real value** in between

Strength

$$s(r | q) = \frac{\Pr(r | q) - \Pr(r)}{1 - \Pr(r)}$$

Strength

$$s(r | q) = \frac{\Pr(r | q) - \Pr(r)}{1 - \Pr(r)}$$

Proportion of **actual** change
to **potential** change

Strength

Properties have probability

Strength

Properties have probability

Bayesian prior / Frequentist occurrences

Strength

Properties have **probability**

Bayesian **prior** / Frequentist **occurrences**

Knowing one **informs** another

Strength

$$s(r | q) = \frac{\Pr(r | q) - \Pr(r)}{1 - \Pr(r)}$$

Strength

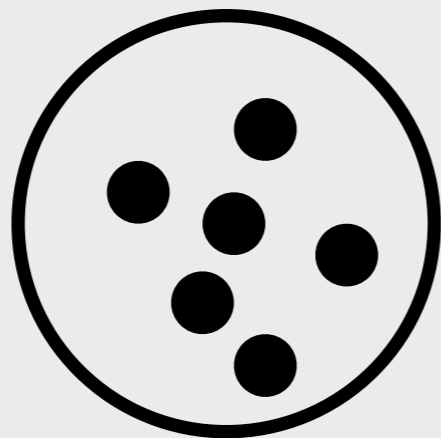
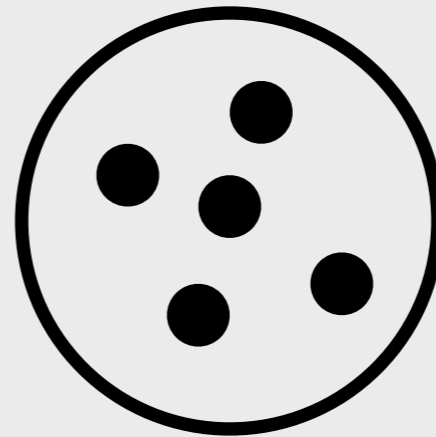
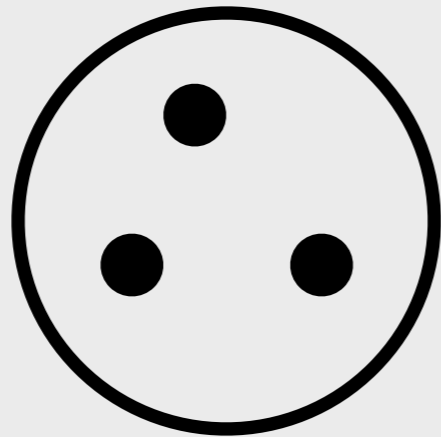
$$s(r | q) = \frac{\Pr(r | q) - \Pr(r)}{1 - \Pr(r)}$$

Proportion of **actual** change
to **potential** change

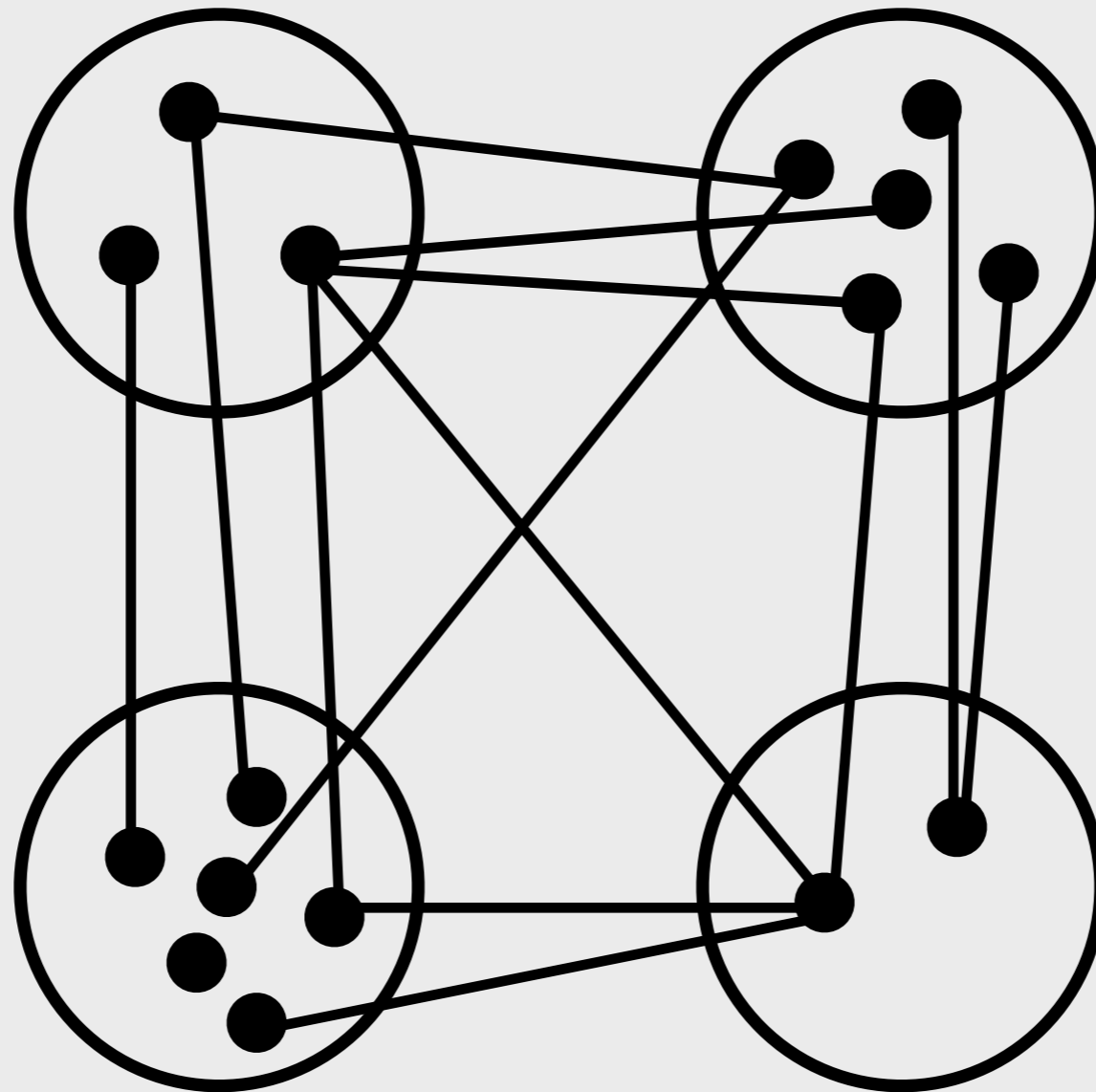
Deriving Correspondences

Derivation

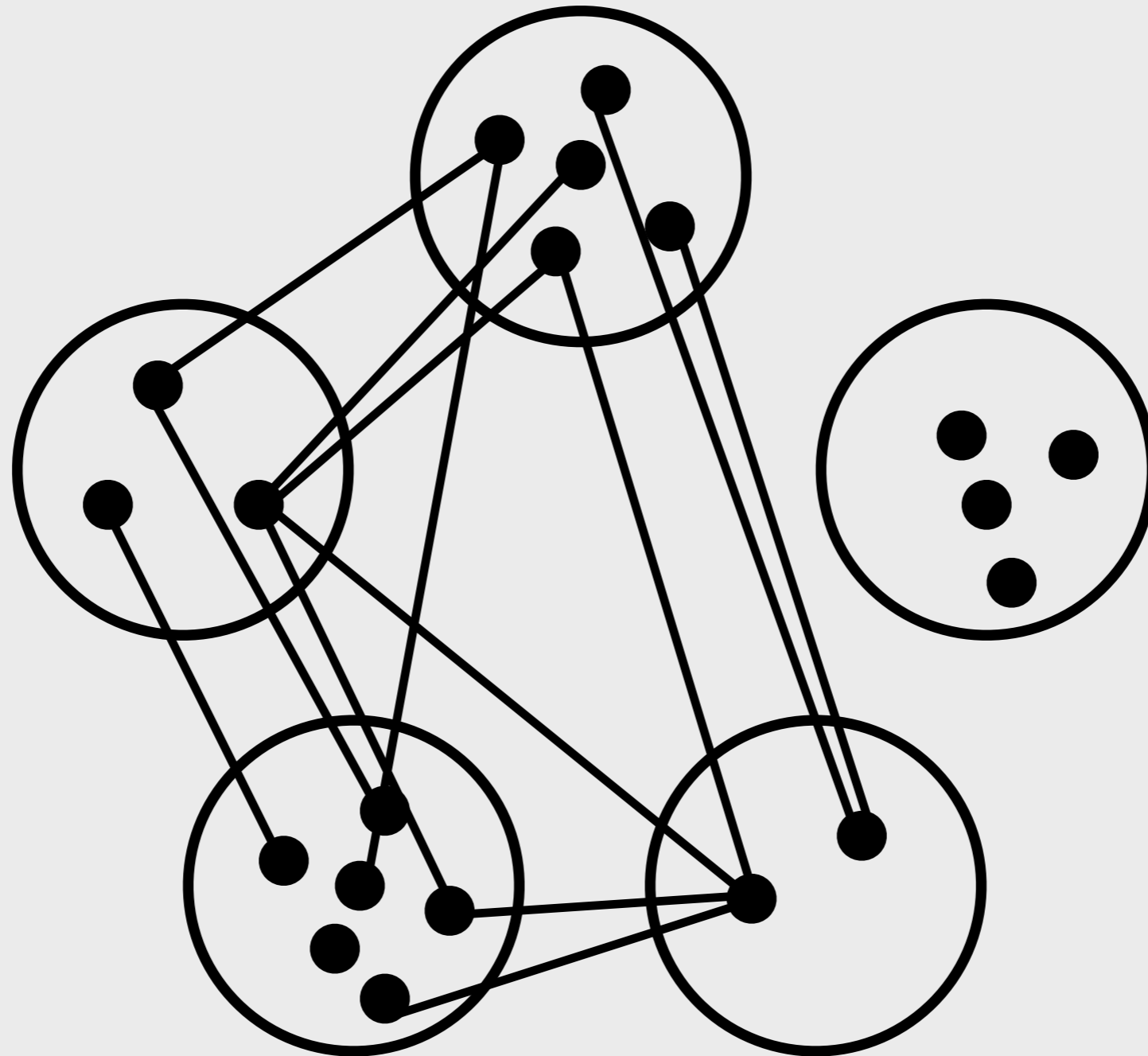
Derivation



Derivation



Derivation



Derivation

Difficult to think of

Derivation

Difficult to think of

Many correspondences

Derivation

Difficult to think of

Many correspondences

More *usually* better

Derivation

Difficult to think of

Many correspondences

More *usually* better



Automate it!

Rules

Rules

$$\overline{\langle a, a, 1 \rangle}$$

If two properties are identical,
they correspond perfectly

Rules

$$\frac{\langle a, b, s \rangle}{\langle b, a, s' \rangle}$$

Correspondences can be
reversed

Rules

$$\frac{\langle a, b, s \rangle}{\langle b, a, s' \rangle}$$

Correspondences can be reversed

$$s' = s \cdot \frac{\Pr(a)}{1 - \Pr(a)} \cdot \frac{1 - \Pr(b)}{\Pr(b)}$$

Rules

$$\frac{\langle a, b, s_1 \rangle \quad \langle c, d, s_2 \rangle}{\langle c[b/a], d, s_1 \cdot s_2 \rangle}$$

Correspondences can be
chained together

Rules

$$\frac{a\{k = v\} \quad b\{k = v'\} \quad \langle a, b, s \rangle}{\langle v, v', s \rangle}$$

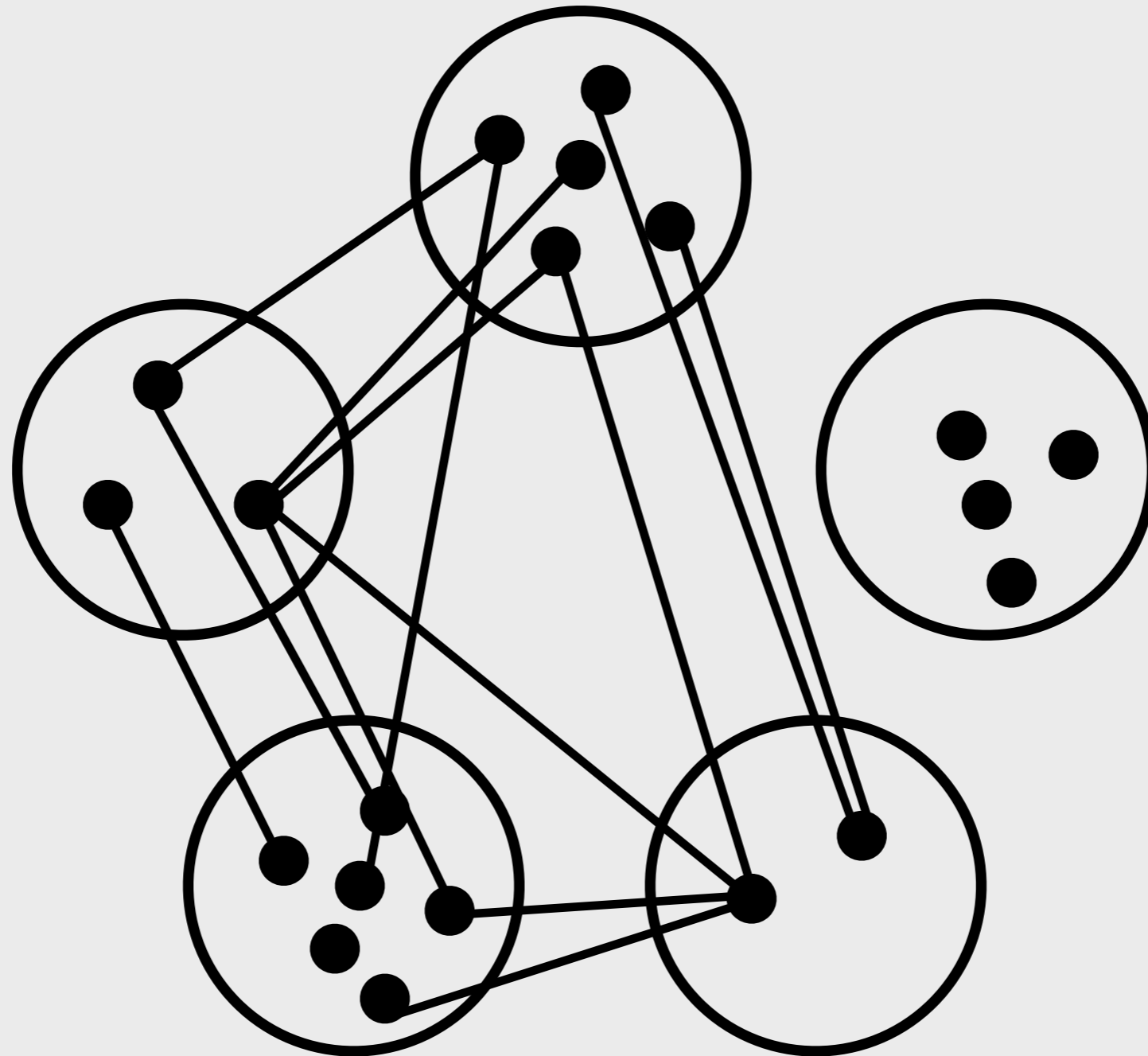
Attributes of corresponding properties
may themselves correspond

Rules

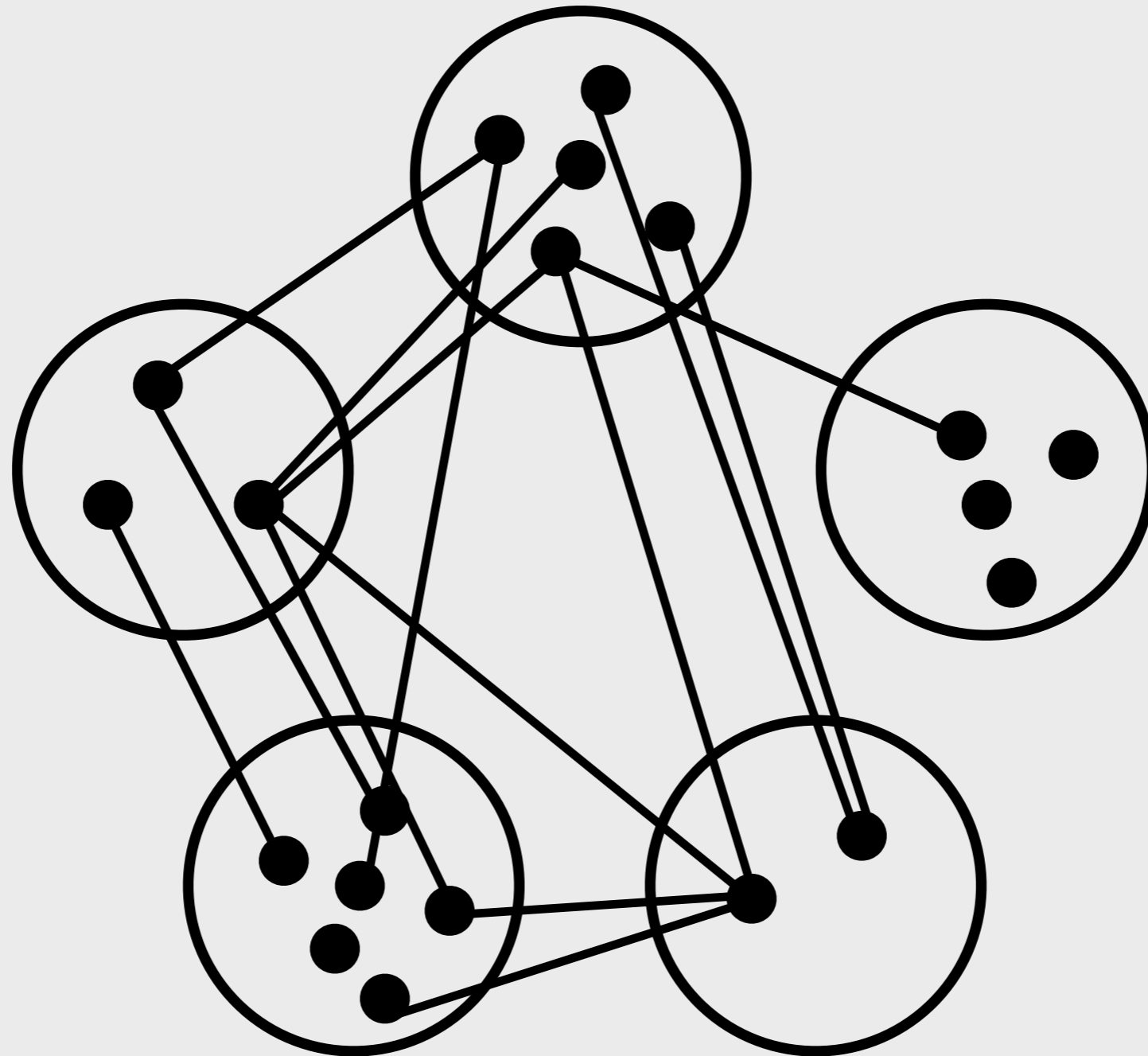
$$\frac{a\{k = v\} \quad b\{k = v'\} \quad \langle v, v', s \rangle}{\langle a, b, s \rangle}$$

Properties with corresponding attributes
may themselves correspond

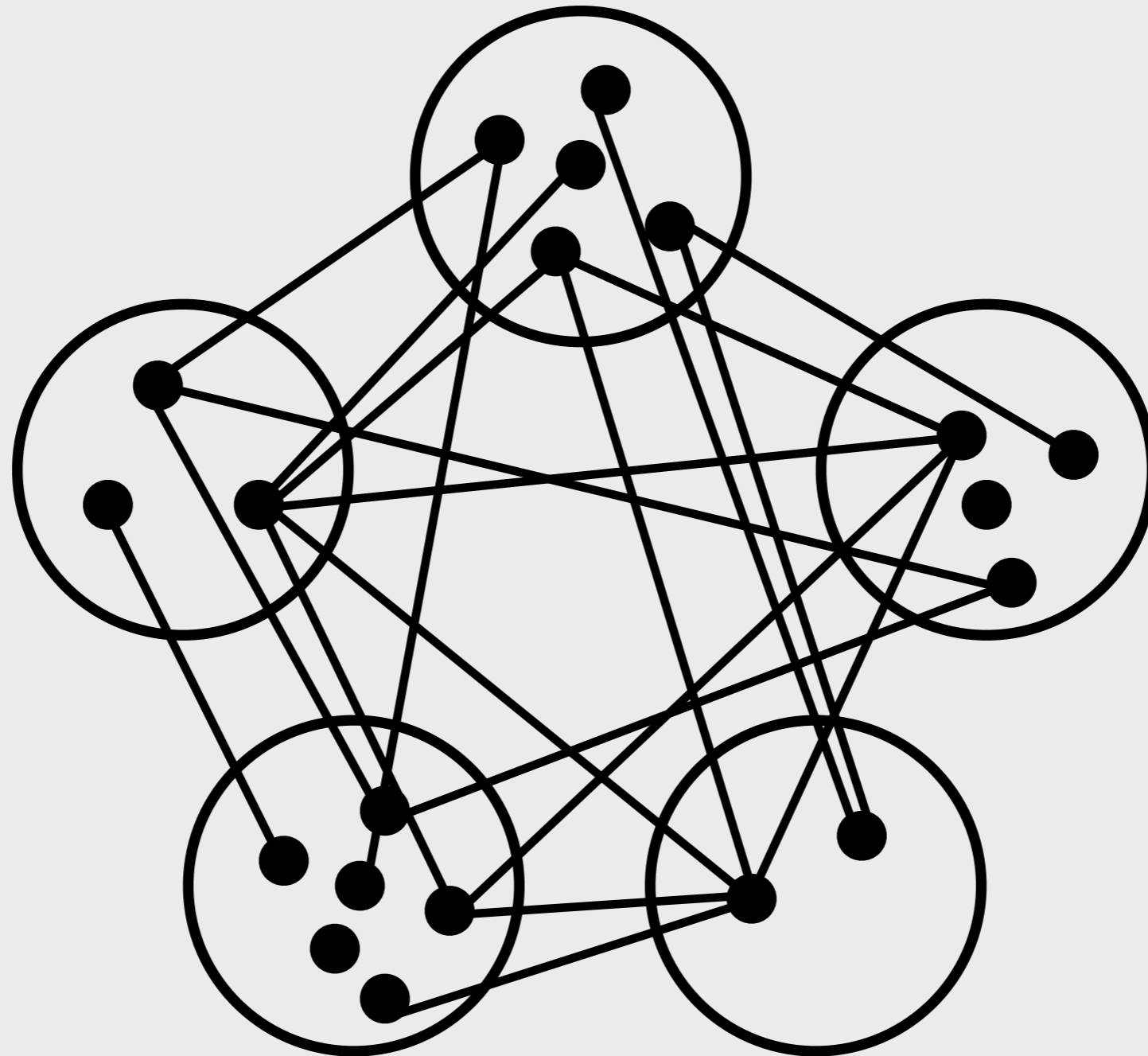
Derivation



Derivation



Derivation



Derivation

$\langle \text{token 1} , \text{token dot} , 1.0 \rangle$

Derivation

⟨ token 1 , token dot , 1.0 ⟩

Derivation

$\langle \text{token 1} , \text{token dot} , 1.0 \rangle$

token 1 : { type = int }



Derivation

$\langle \text{token 1} , \text{token dot} , 1.0 \rangle$

token 1 : { type = int }

token dot : { type = arrangement }

Derivation

$\langle \text{token 1} , \text{token dot} , 1.0 \rangle$

token 1 : { type = int }

token dot : { type = arrangement }

Derivation

$\langle \text{token 1} , \text{token dot} , 1.0 \rangle$

token 1 : { type = int }

token dot : { type = arrangement }

$\langle \text{type int} , \text{type arrangement} , 1.0 \rangle$

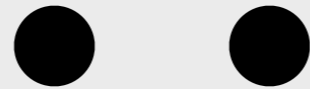
Domination

Domination

Multiple derivations

Domination

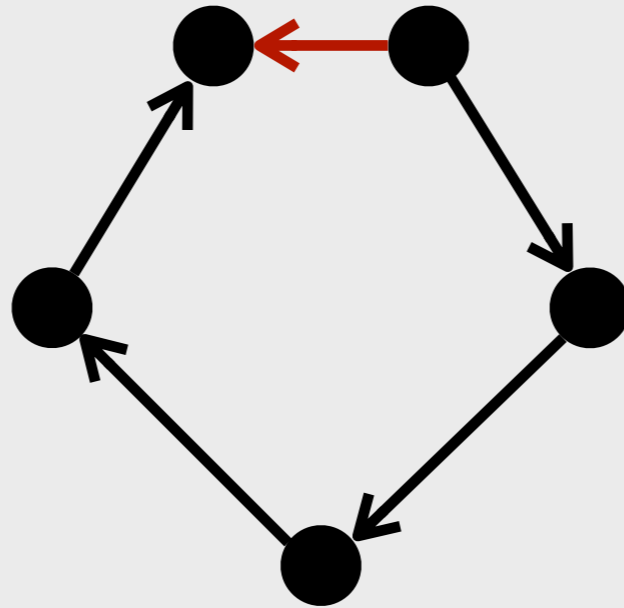
Domination



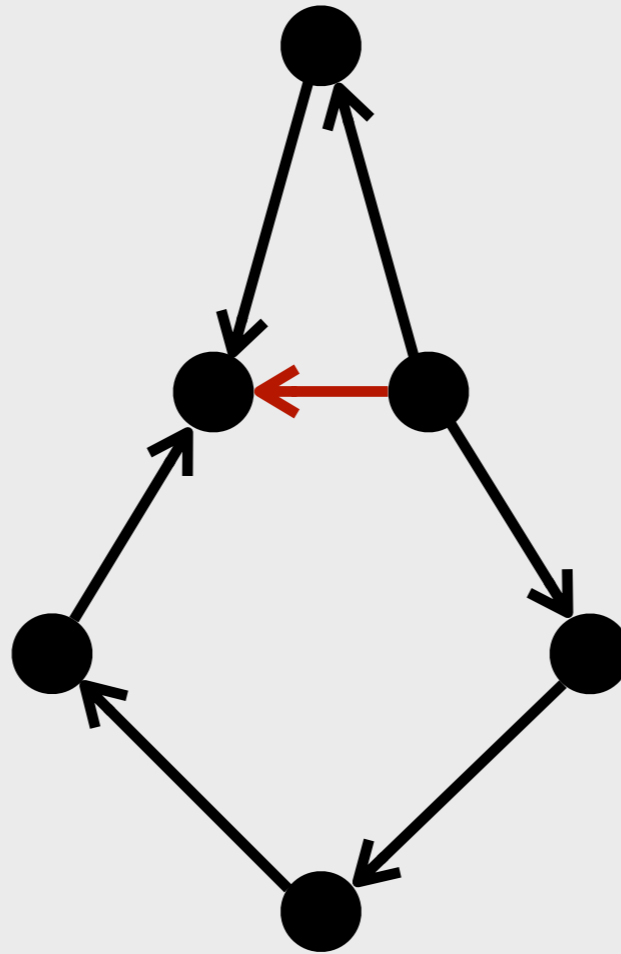
Domination



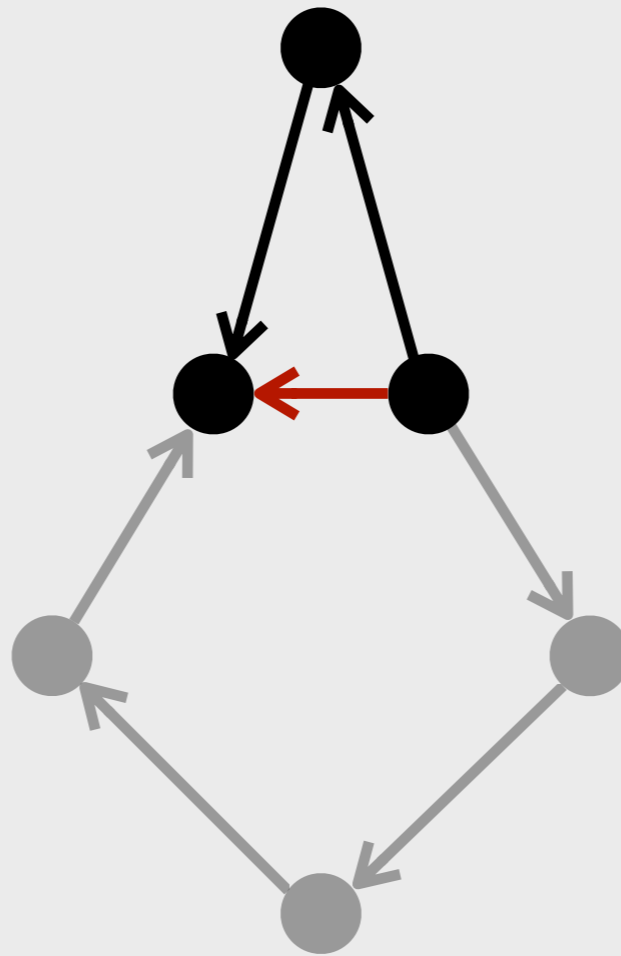
Domination



Domination



Domination



Domination

Multiple derivations

Domination

Multiple **derivations**

More **specific** rules, or **stronger** rules,
dominate

Domination

Multiple **derivations**

More **specific** rules, or **stronger** rules,
dominate

How to **order**?

Domination

Multiple **derivations**

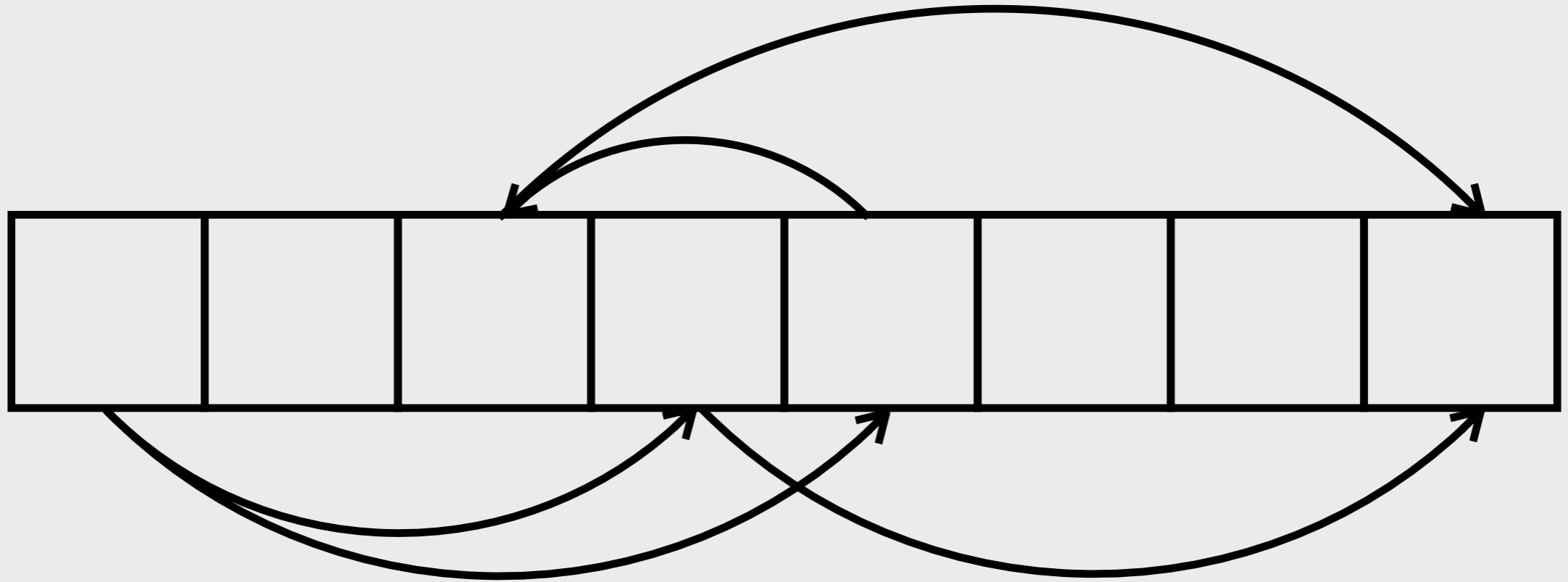
More **specific** rules, or **stronger** rules,
dominate

How to **replace**?

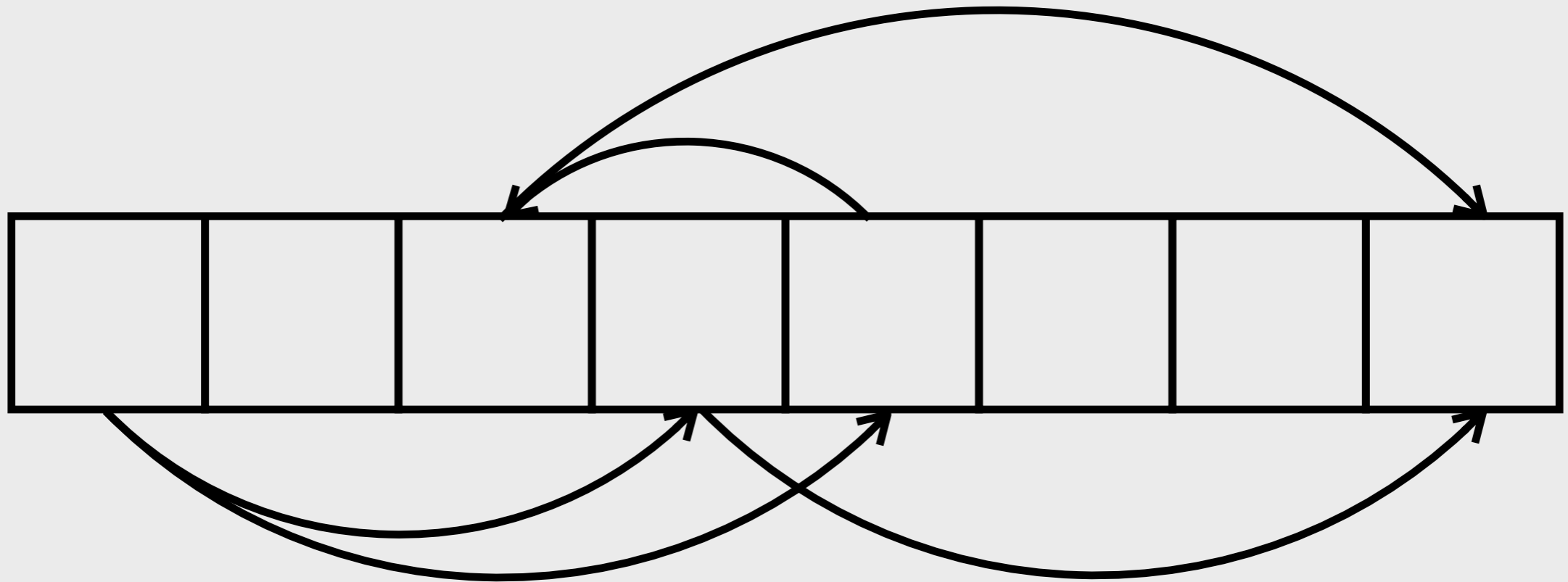
Domination

--	--	--	--	--	--	--	--

Domination

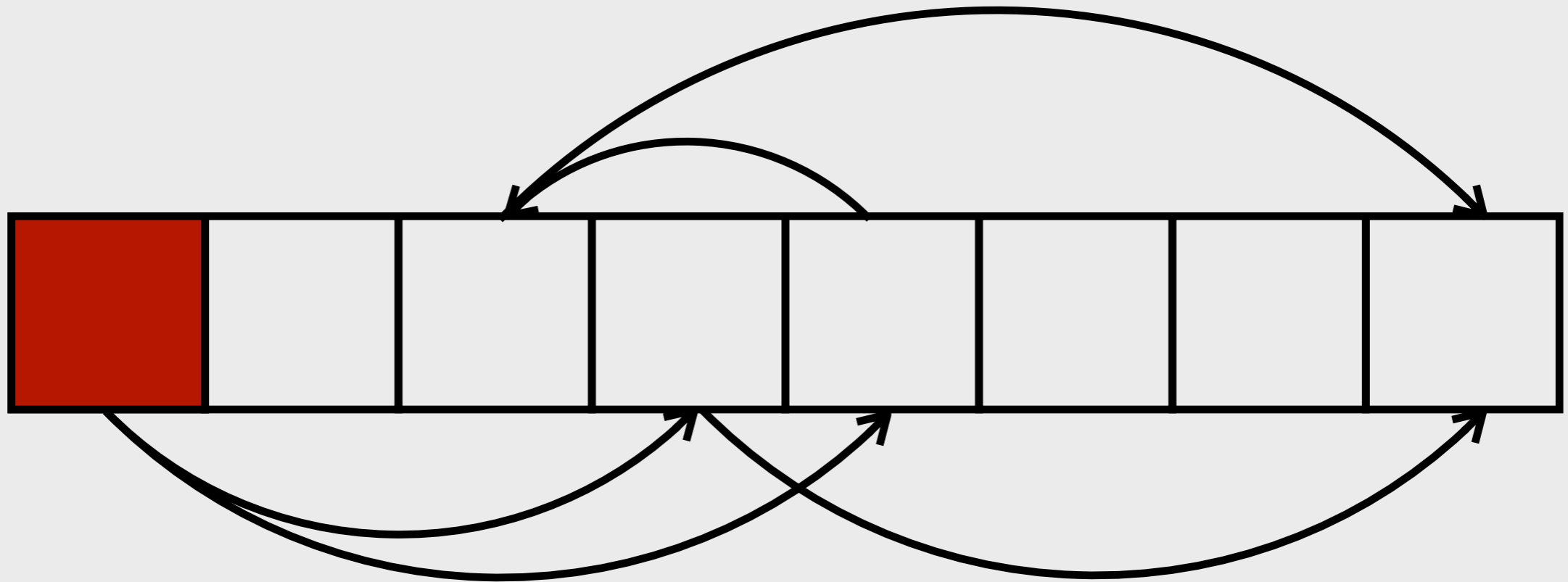


Domination



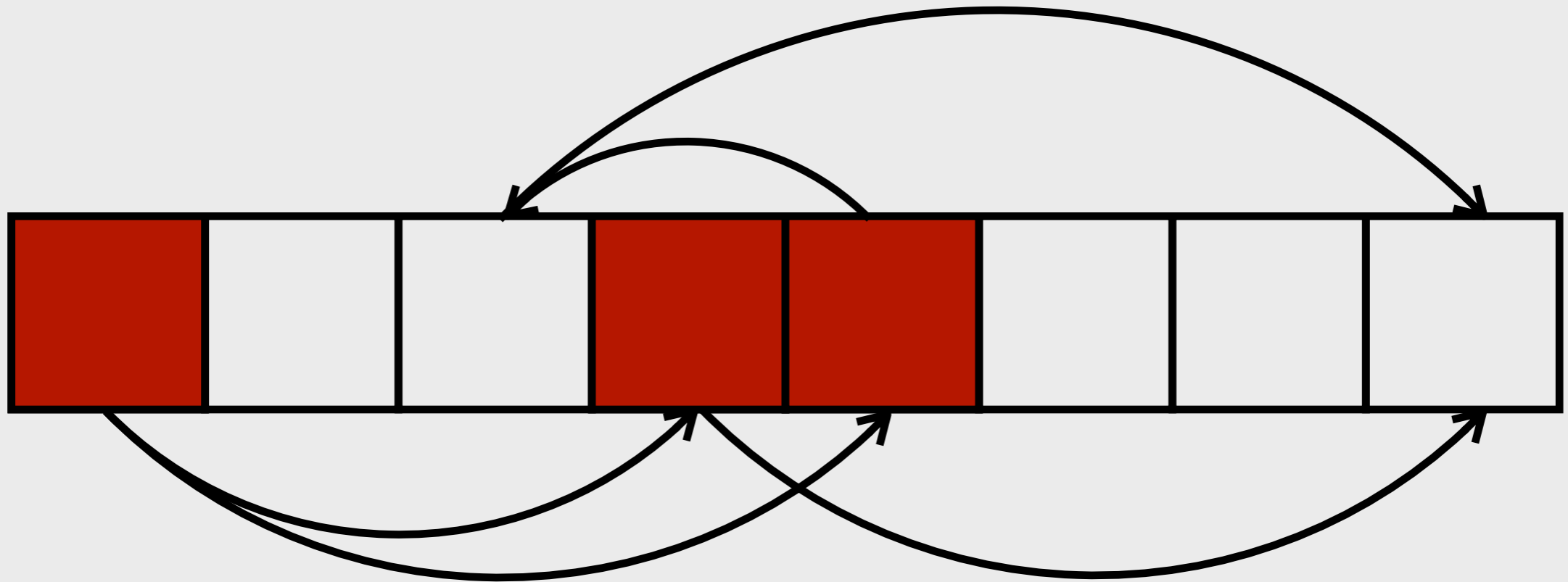
Derivation children

Domination



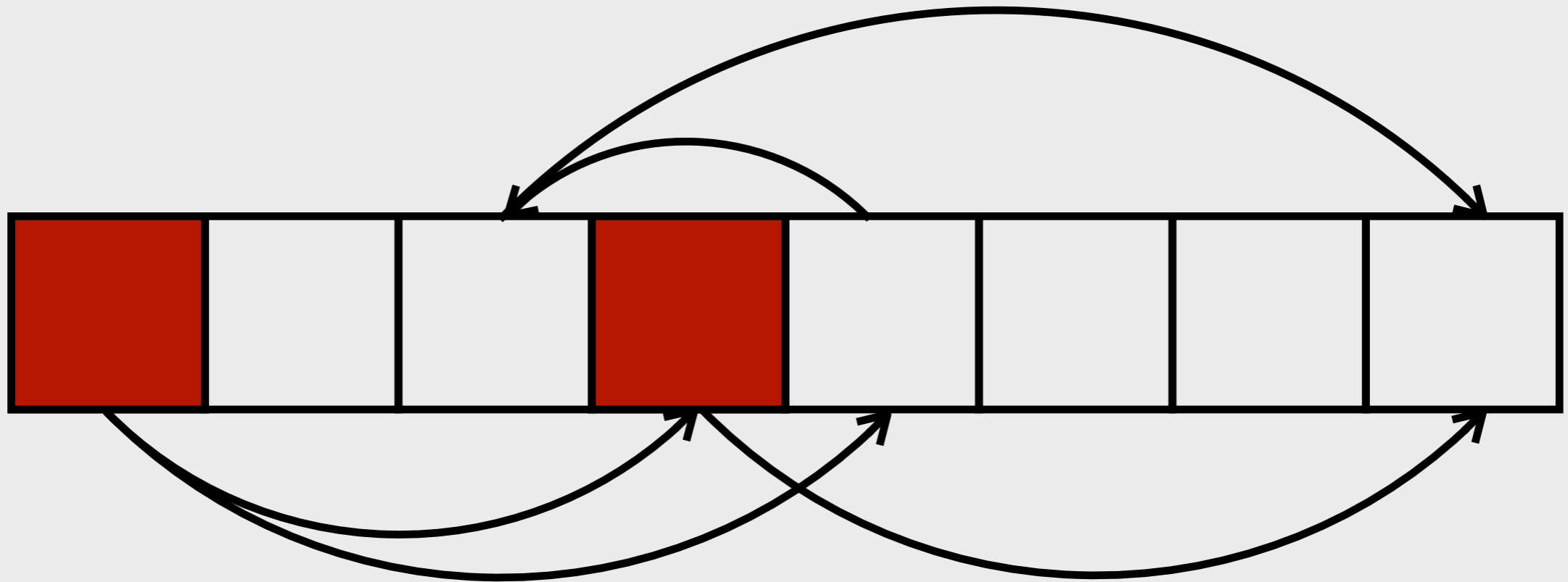
Derivation children

Domination



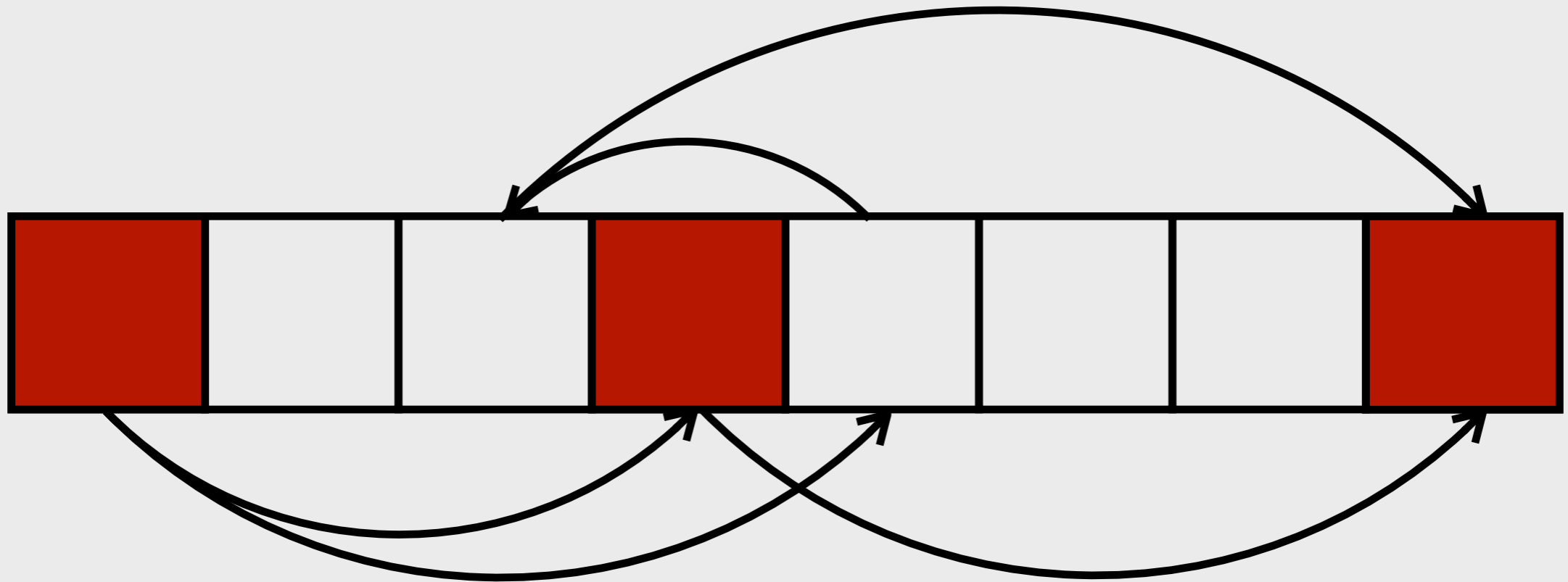
Derivation children

Domination



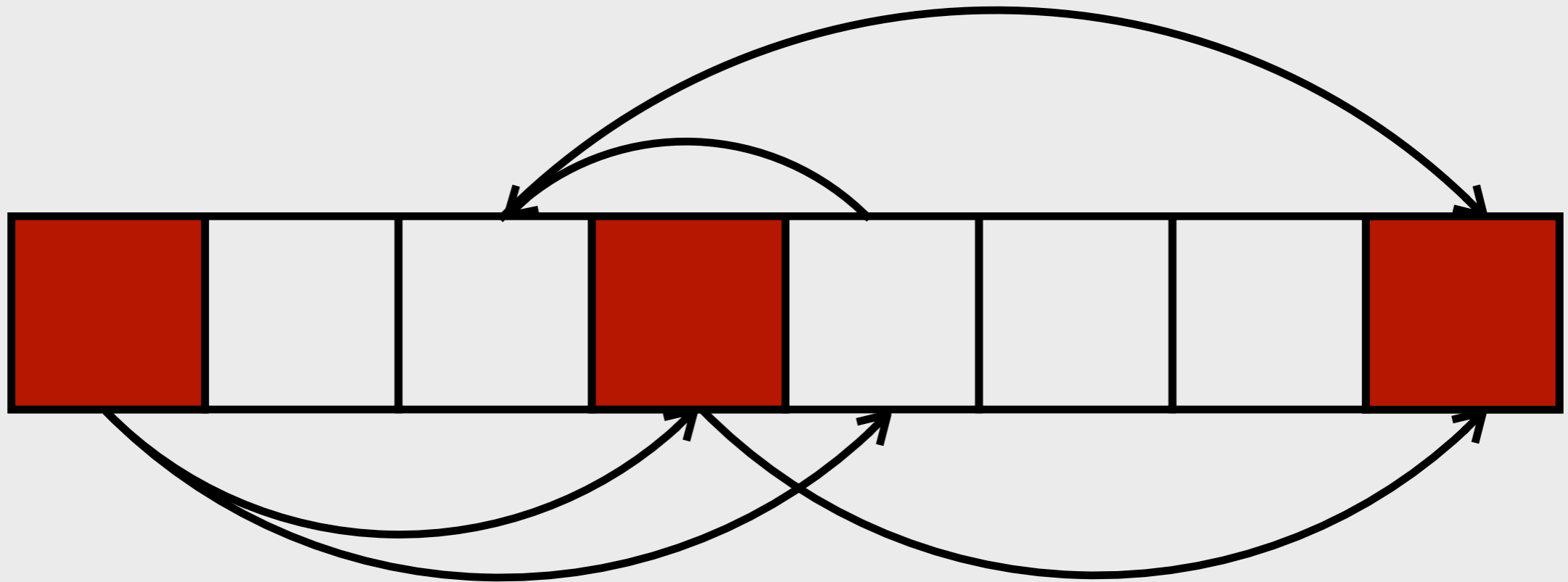
Derivation children

Domination



Derivation children

Domination



Derivation children

Enforce acyclicity (to avoid infinite loops!)

Generalising

Generalising

Recommend other things

Generalising

Recommend other things

Films, books, music

Generalising

Recommend other things

Films, books, music → all together

Generalising

Recommend other things

Films, books, music → all together

Post-hoc rationalisation

Generalising

Set of structures \mathcal{S}

Generalising

Set of structures \mathcal{S}

Each $S \in \mathcal{S}$ is a tuple $S = (A, \mathcal{R}, \text{Pr})$

Generalising

Set of structures \mathcal{S}

Each $S \in \mathcal{S}$ is a tuple $S = (A, \mathcal{R}, \text{Pr})$

Set A contains atoms

Generalising

Set of structures \mathcal{S}

Each $S \in \mathcal{S}$ is a tuple $S = (A, \mathcal{R}, \text{Pr})$

Set A contains atoms

Set \mathcal{R} contains predicates on $A^2 \dots A^n$

Generalising

Set of structures \mathcal{S}

Each $S \in \mathcal{S}$ is a tuple $S = (A, \mathcal{R}, \text{Pr})$

Set A contains atoms

Set \mathcal{R} contains predicates on $A^2 \dots A^n$

Function Pr assigns probabilities to atoms

Generalising

Generalising

$$\overline{\langle a, a, 1 \rangle}$$

Generalising

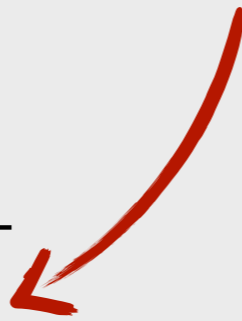
$$\frac{\langle a, a, 1 \rangle}{\langle a, b, s \rangle} \quad \frac{\langle a, b, s \rangle}{\langle b, a, s' \rangle}$$

Generalising

Reversal formula earlier

$$\frac{}{\langle a, a, 1 \rangle}$$

$$\frac{\langle a, b, s \rangle}{\langle b, a, s' \rangle}$$



Generalising

Reversal formula earlier


$$\frac{}{\langle a, a, 1 \rangle}$$

$$\frac{\langle a, b, s \rangle}{\langle b, a, s' \rangle}$$

$$\frac{\langle a, b, s_1 \rangle \quad \langle c, d, s_2 \rangle}{\langle c[b/a], d, s_1 \cdot s_2 \rangle}$$

Generalising

Reversal formula earlier

$$\frac{\langle a, a, 1 \rangle}{\langle a, b, s \rangle} \quad \frac{\langle a, b, s \rangle}{\langle b, a, s' \rangle} \quad \frac{\langle a, b, s_1 \rangle \quad \langle c, d, s_2 \rangle}{\langle c[b/a], d, s_1 \cdot s_2 \rangle}$$


$$\frac{P(x_1, \dots, x_n) \quad P(y_1, \dots, y_n) \quad \langle x_1, y_1, s_1 \rangle \cdots \langle x_k, y_k, s_k \rangle}{\langle x_{k+1}, y_{k+1}, s' \rangle \cdots \langle x_n, y_n, s' \rangle}$$

Generalising

Reversal formula earlier

$$\frac{\langle a, a, 1 \rangle}{\langle a, b, s \rangle} \quad \frac{\langle a, b, s \rangle}{\langle b, a, s' \rangle} \quad \frac{\langle a, b, s_1 \rangle \quad \langle c, d, s_2 \rangle}{\langle c[b/a], d, s_1 \cdot s_2 \rangle}$$

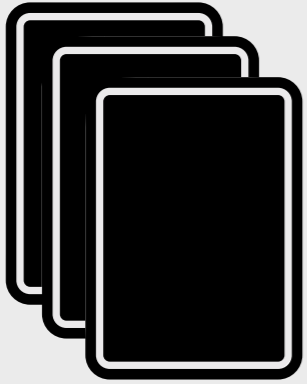
$$\frac{P(x_1, \dots, x_n) \quad P(y_1, \dots, y_n) \quad \langle x_1, y_1, s_1 \rangle \cdots \langle x_k, y_k, s_k \rangle}{\langle x_{k+1}, y_{k+1}, s' \rangle \cdots \langle x_n, y_n, s' \rangle}$$

Where $s' = \frac{1}{n-1} \sum_{i=1}^k s_i$

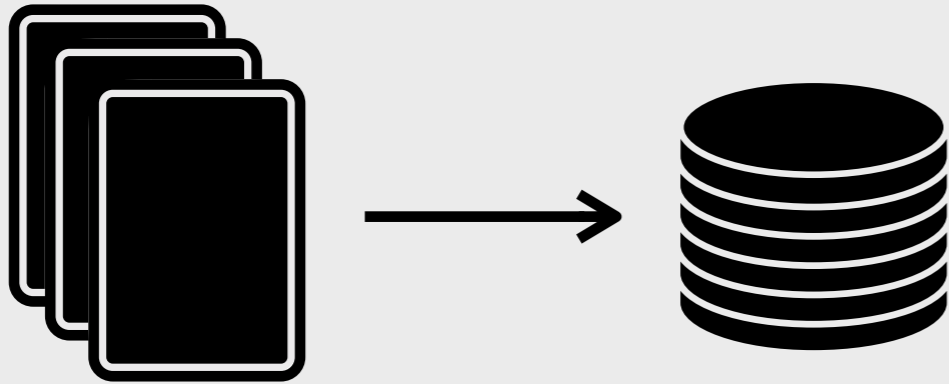
In the rep2rep Framework

rep2rep

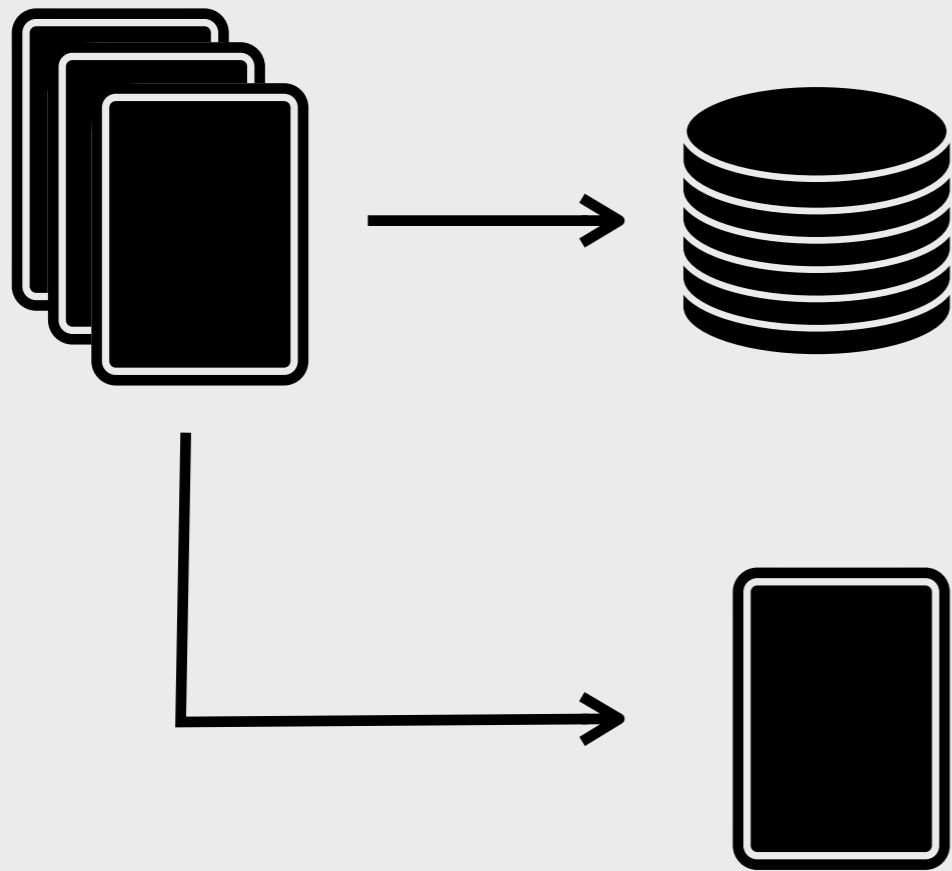
rep2rep



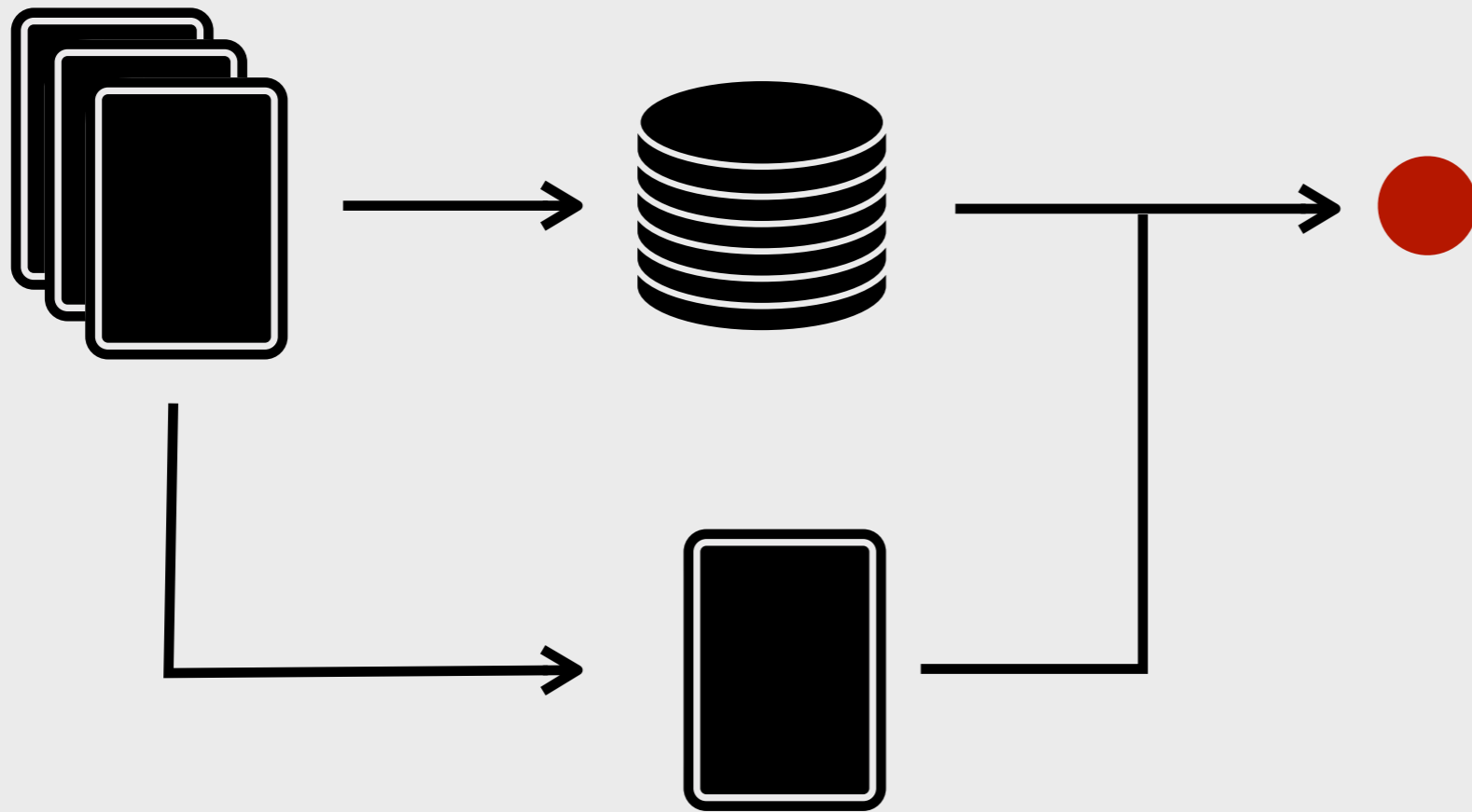
rep2rep



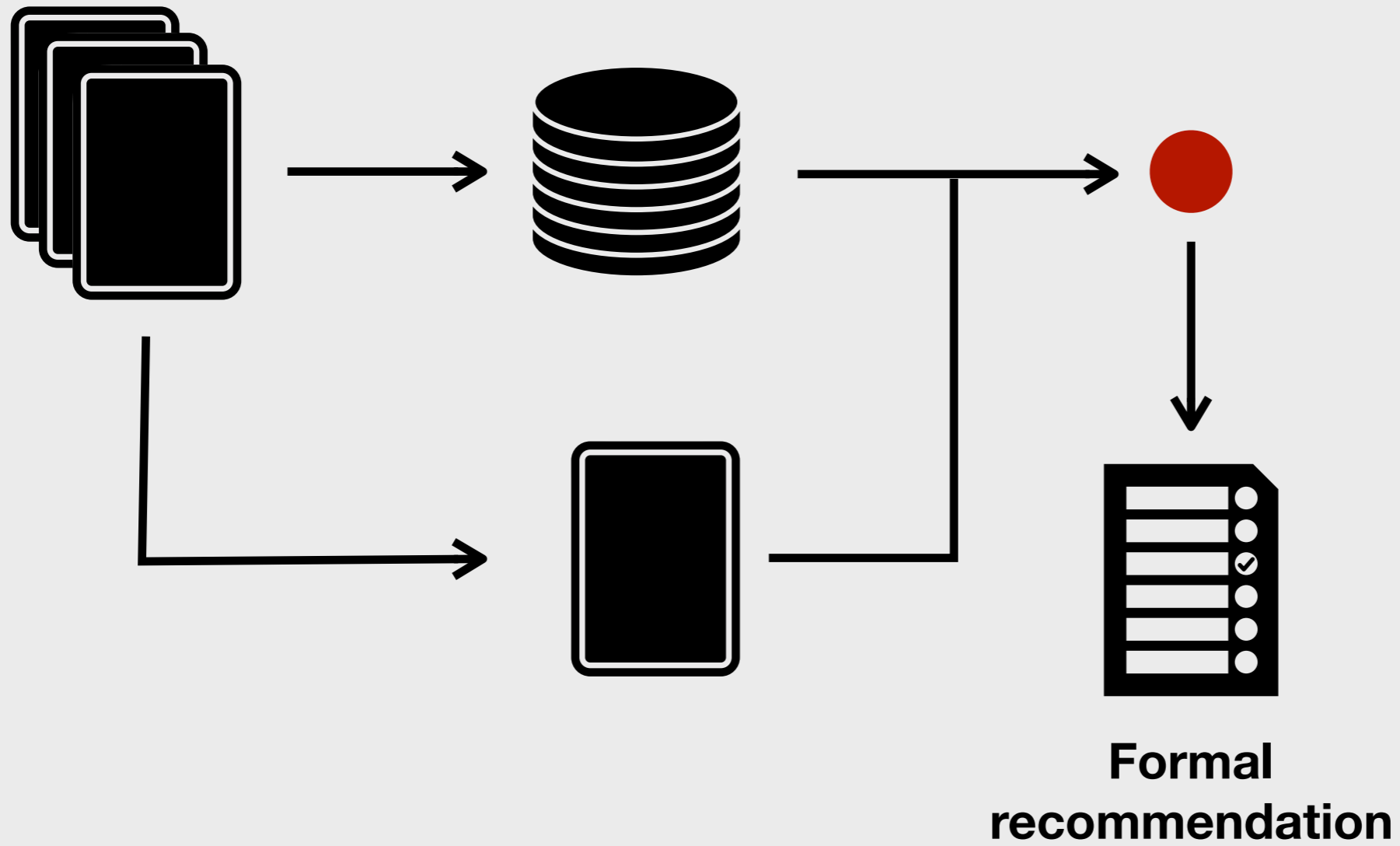
rep2rep



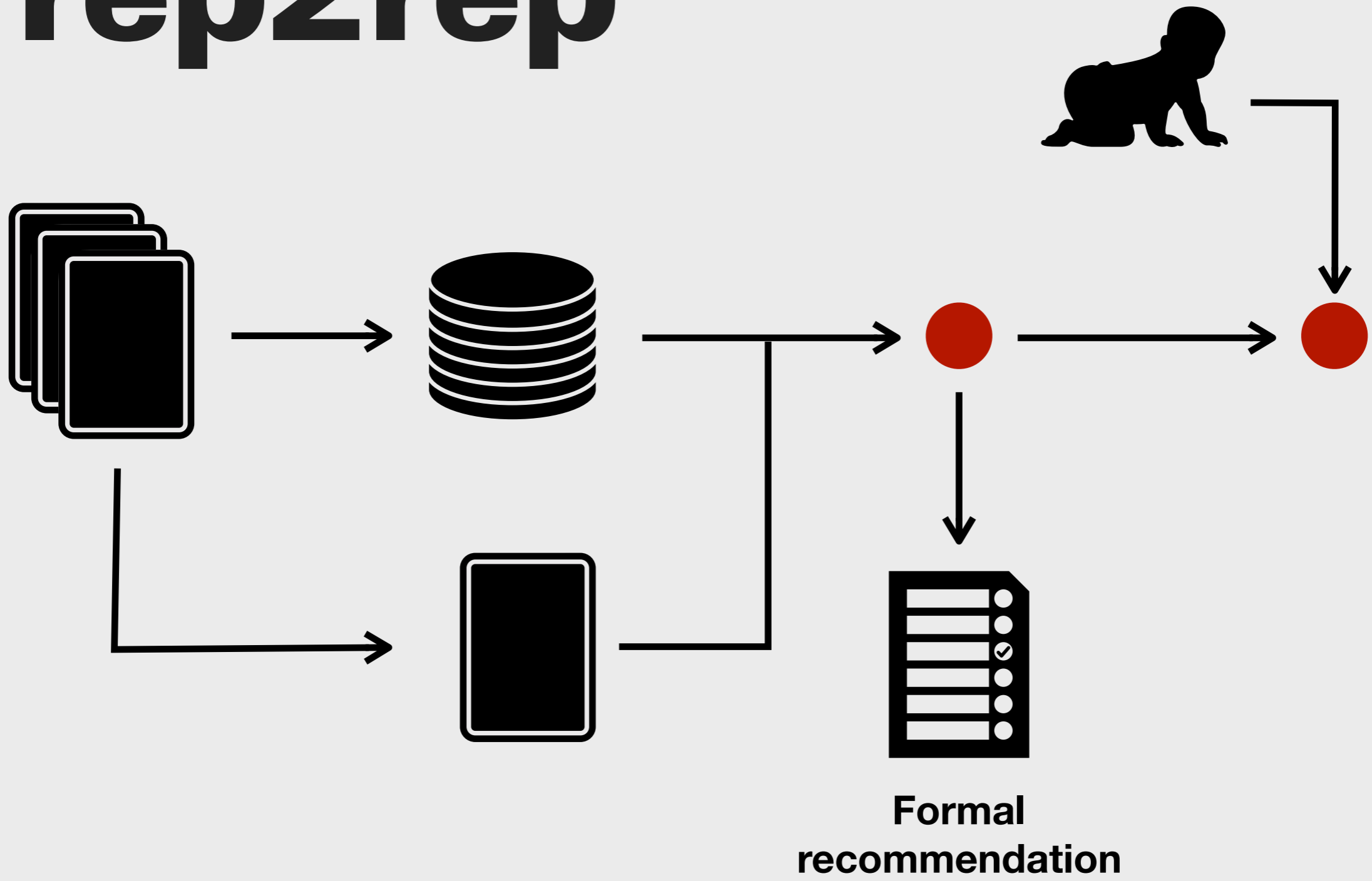
rep2rep



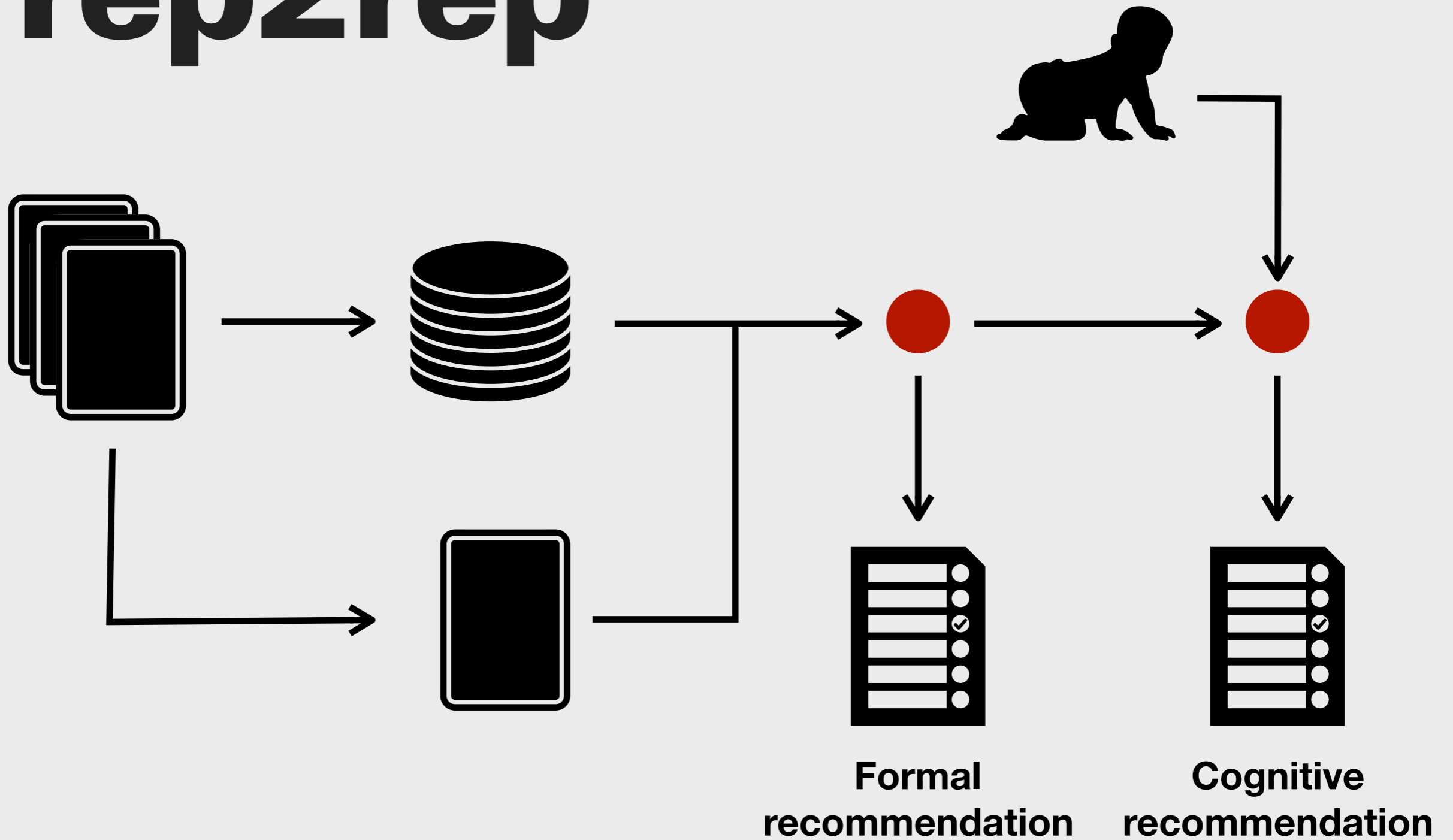
rep2rep



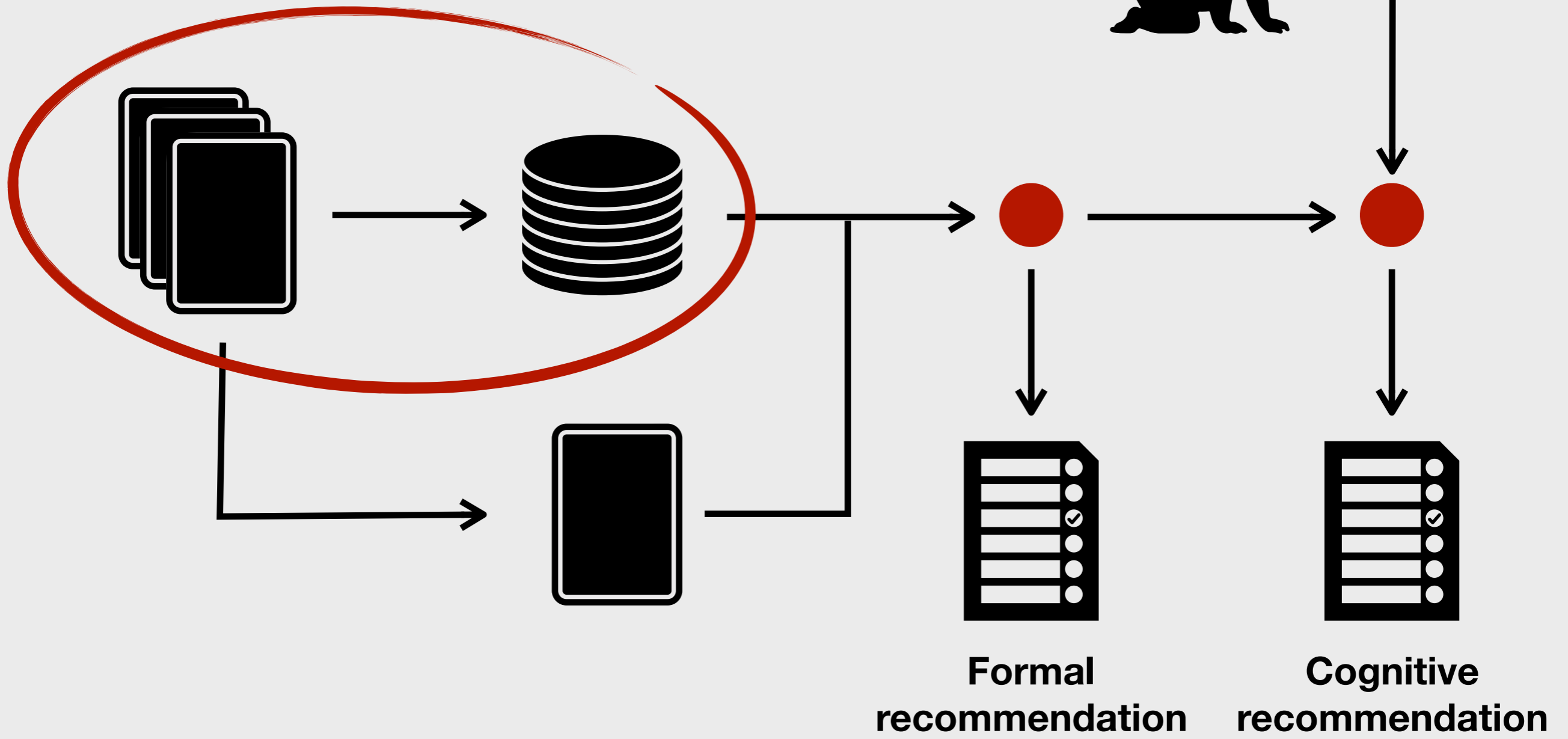
rep2rep



rep2rep



rep2rep



Finding an Analogy

Aaron Stockdill

Mateja Jamnik, Daniel Raggi

Peter Cheng, Grecia Garcia Garcia, Holly Sutherland