What is a number, and what should it be?

Richard Dedekind

Finding an Analogy

Aaron Stockdill

Mateja Jamnik, Daniel Raggi
Peter Cheng, Grecia Garcia Garcia, Holly Sutherland

Gaussian Sum

Gaussian Sum

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

Gaussian Sum

$$\sum_{i=1}^{n} i =$$

Gaussian Sum

$$\sum_{i=1}^{n} i = 1 + 2 + \cdots + n$$

Gaussian Sum

$$\sum_{i=1}^{n} i = 1 + 2 + \cdots + n$$
$$= n + \cdots + 2 + 1$$

Gaussian Sum

$$\sum_{i=1}^{n} i = 1 + 2 + \cdots + n$$
$$= n + \cdots + 2 + 1$$

Gaussian Sum

$$\begin{aligned}\sum_{i=1}^{n} i &= 1 + 2 + \cdots + n \\ &= n + \cdots + 2 + 1 \\ &= \frac{1}{2} \times n \times (n+1)\end{aligned}$$

Gaussian Sum

$$\sum_{i=1}^{n} i = 1 + 2 + \cdots + n$$
$$= n + \cdots + 2 + 1$$
$$= \frac{n(n+1)}{2}$$

Gaussian Sum

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

Gaussian Sum

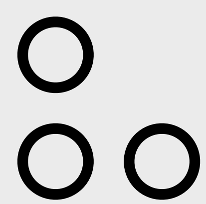
$$\sum_{i=1}^{n} i =$$

Gaussian Sum

$$\sum_{i=1}^{n} i =$$

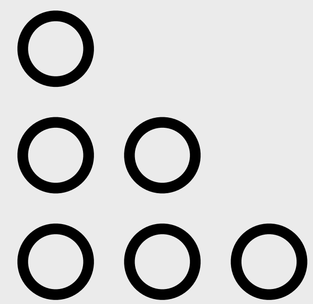
Gaussian Sum

$$\sum_{i=1}^{n} i =$$



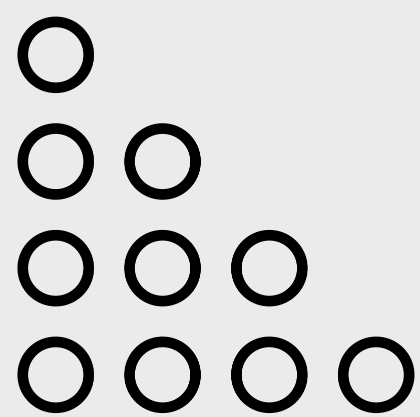
Gaussian Sum

$$\sum_{i=1}^{n} i =$$



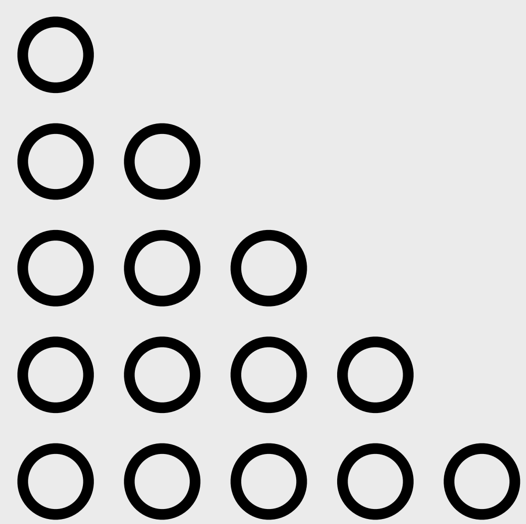
Gaussian Sum

$$\sum_{i=1}^{n} i =$$



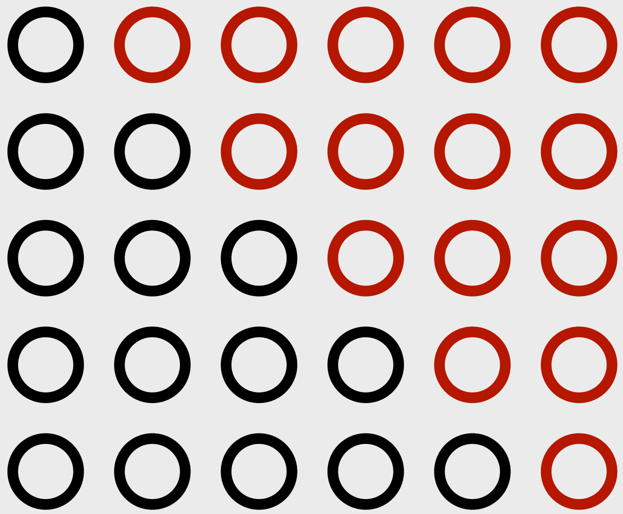
Gaussian Sum

$$\sum_{i=1}^{n} i =$$



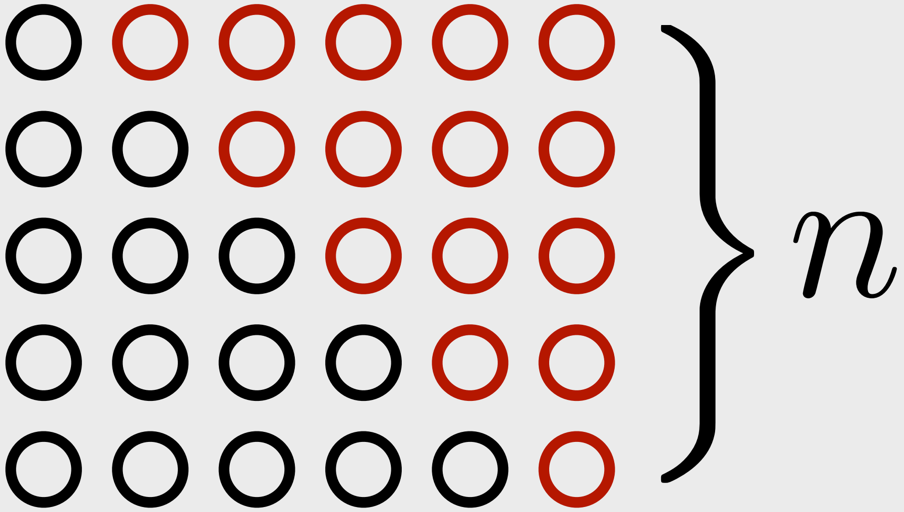
Gaussian Sum

$$\sum_{i=1}^{n} i =$$



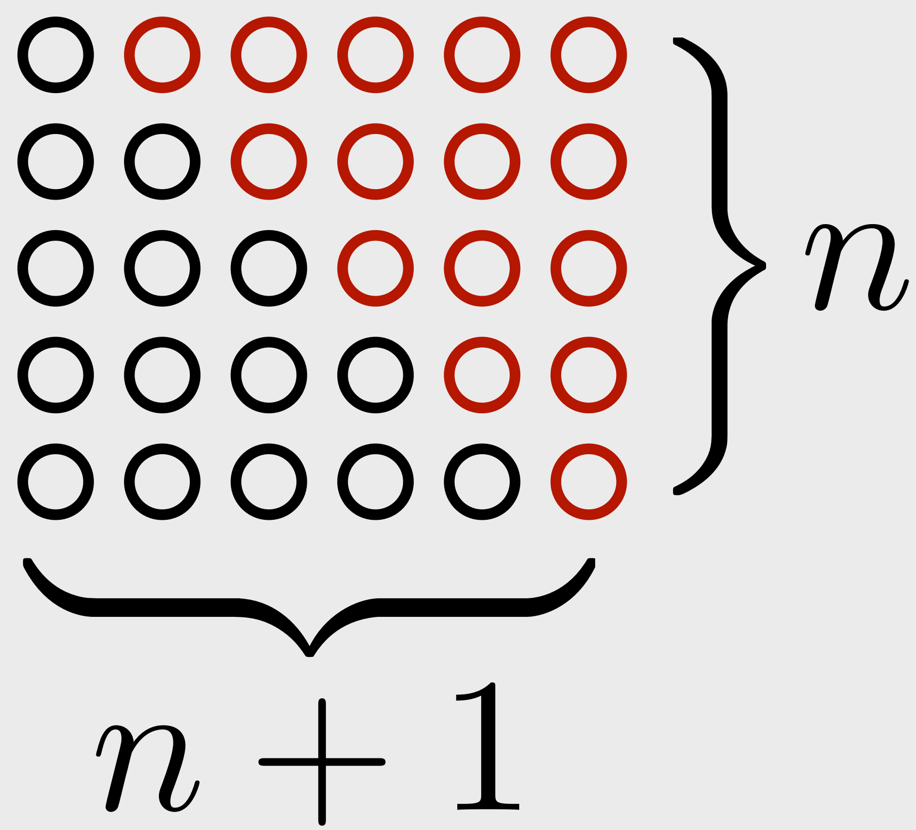
Gaussian Sum

$$\sum_{i=1}^{n} i =$$

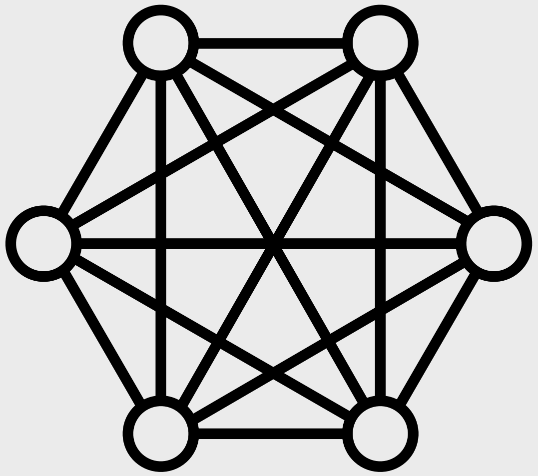


Gaussian Sum

$$\sum_{i=1}^{n} i =$$

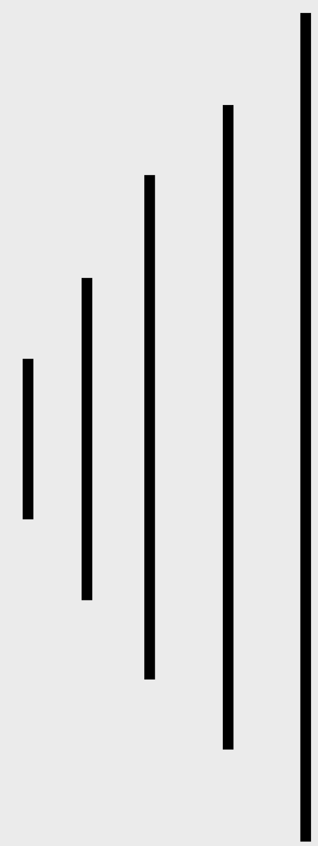


Gaussian Sum

$$\sum_{i=1}^{n} i =$$ 

Gaussian Sum

$$\sum_{i=1}^{n} i =$$



Why?

Why?

People think differently

Why?

People think differently

Same content, different cognitive features

Why?

People think differently

Same* content, different cognitive features

Why?

People think differently

Same content, different cognitive features

Switch to a more accessible representation

Why?

People think differently

Same content, different cognitive features

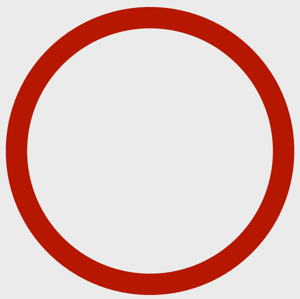
Switch to a more accessible representation

Expose conceptual links

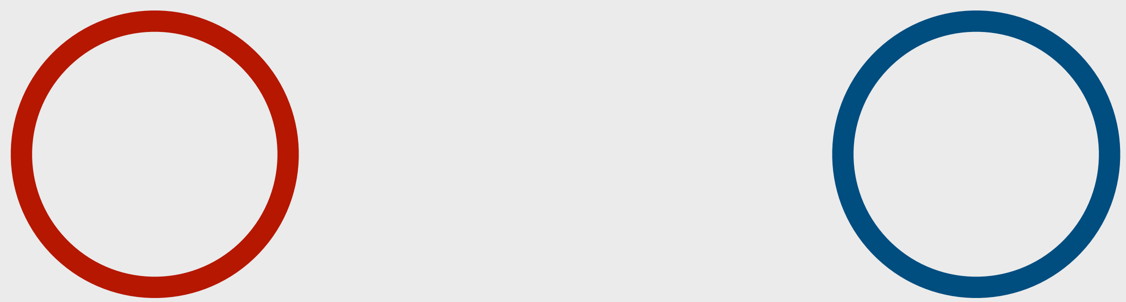
Structure Mapping

Structure Maps

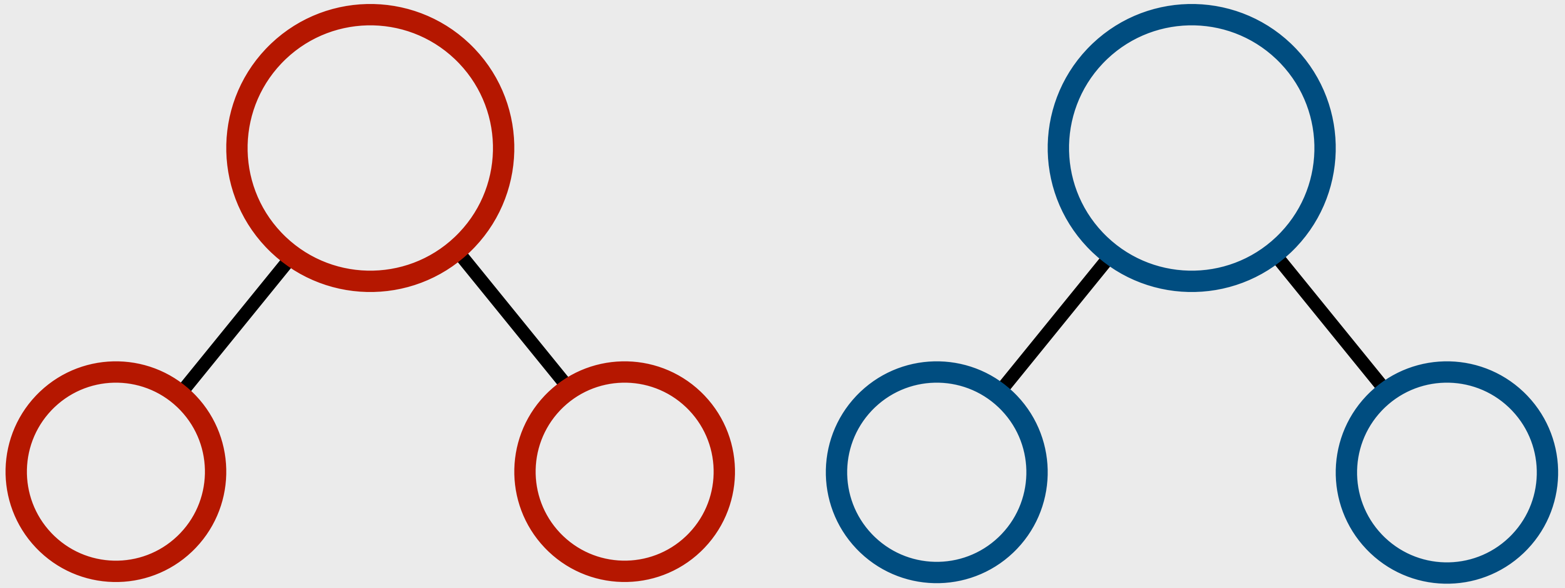
Structure Maps



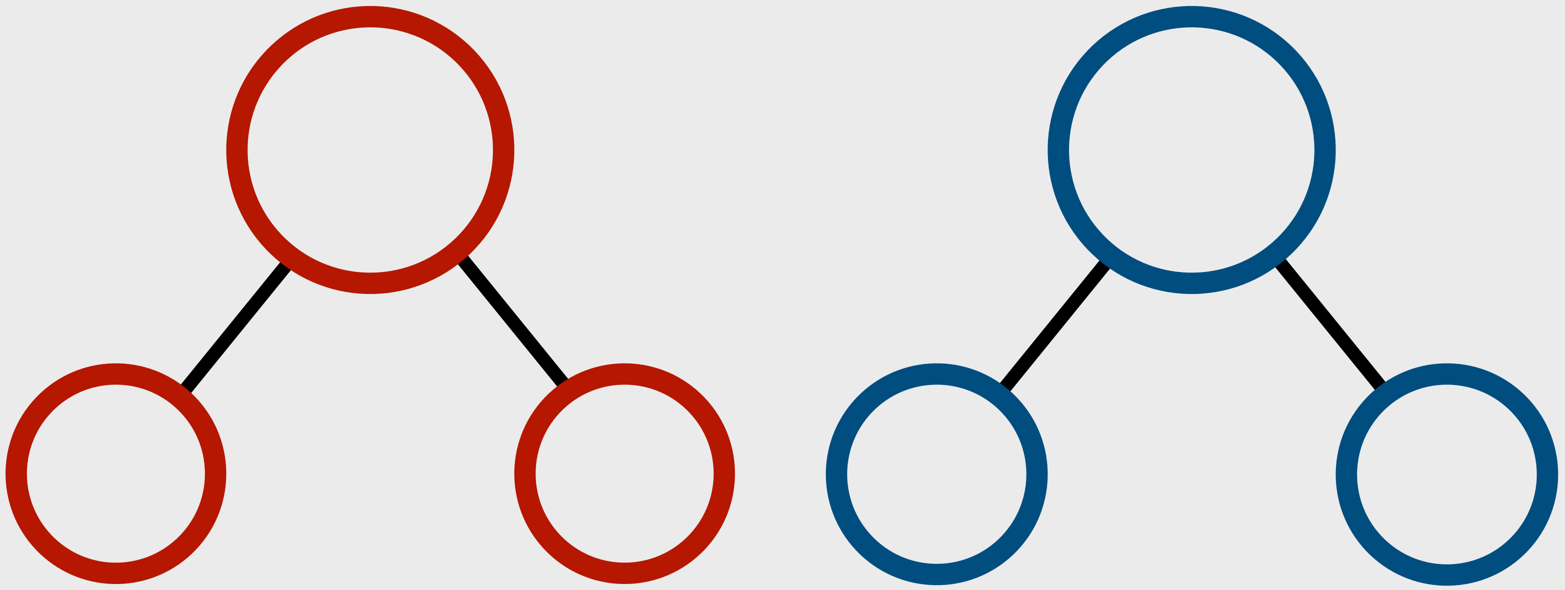
Structure Maps



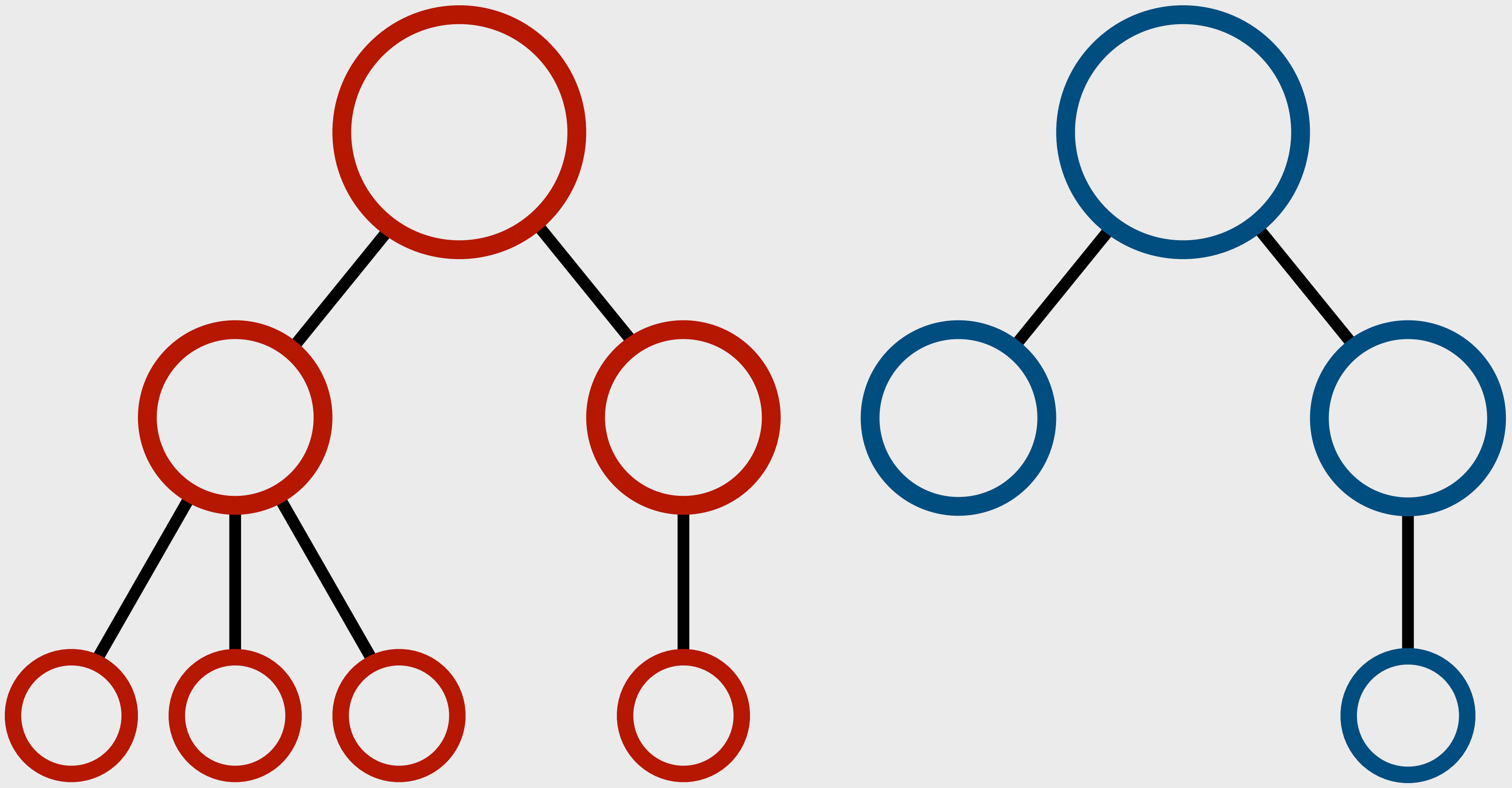
Structure Maps



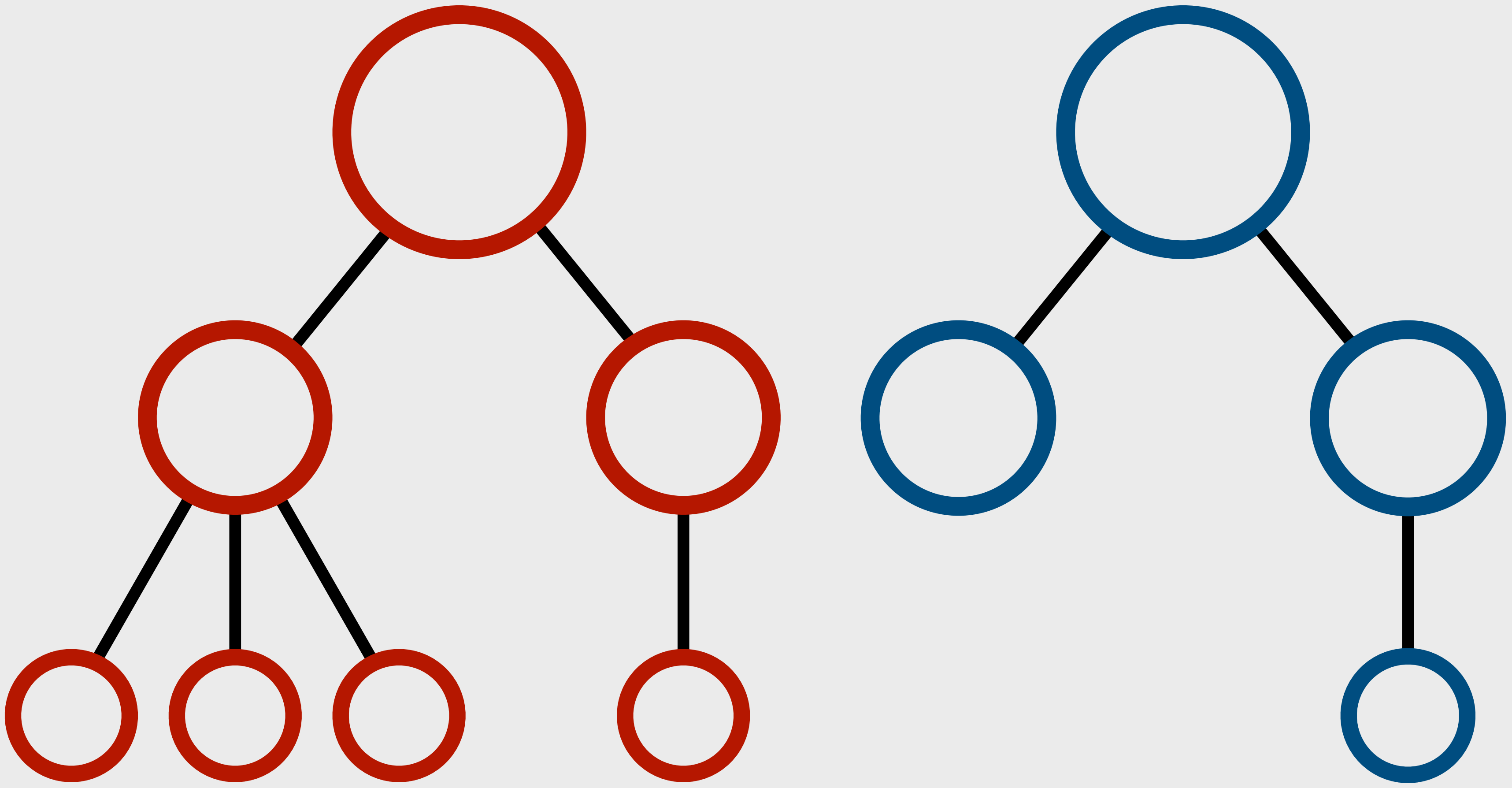
Structure Maps



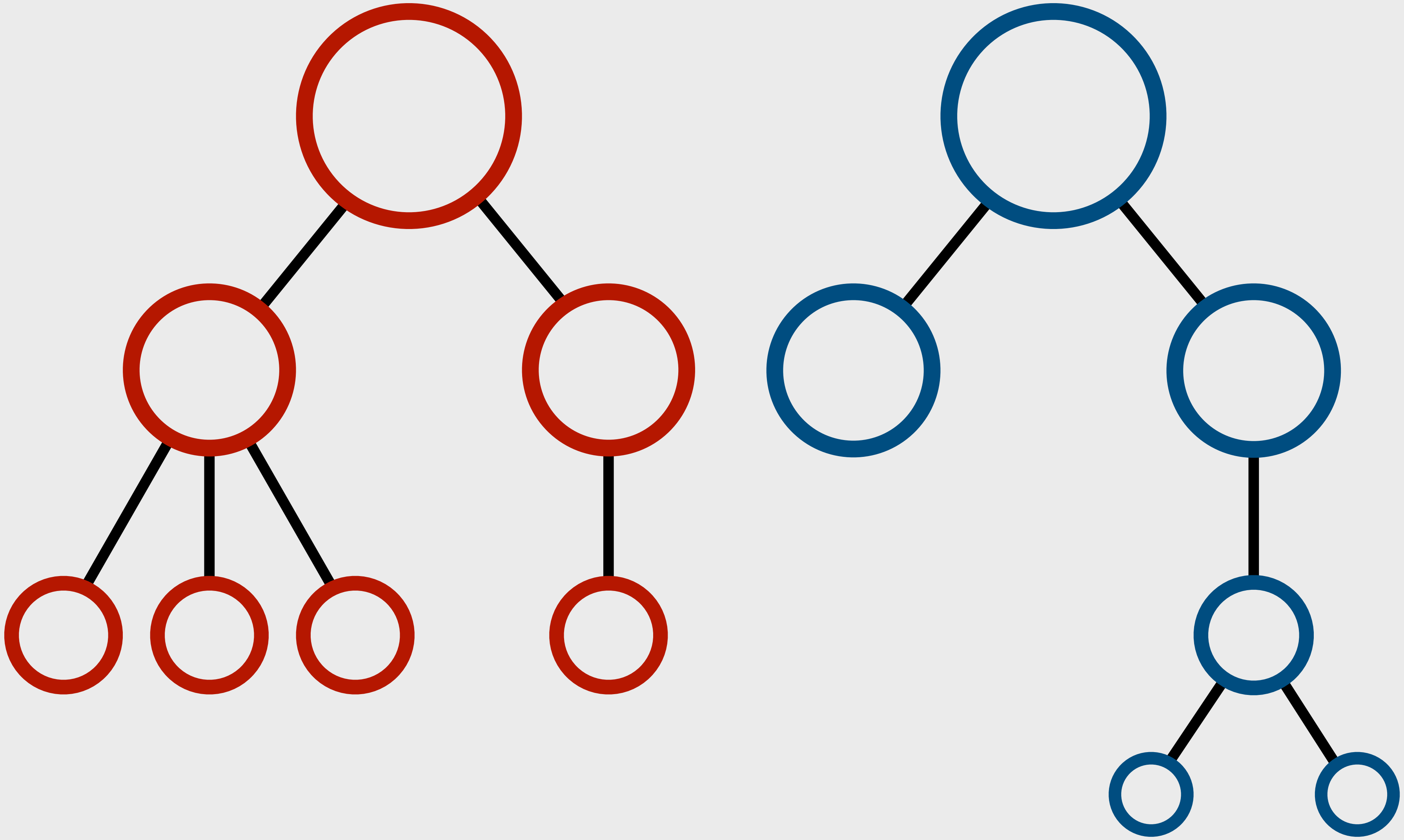
Structure Maps



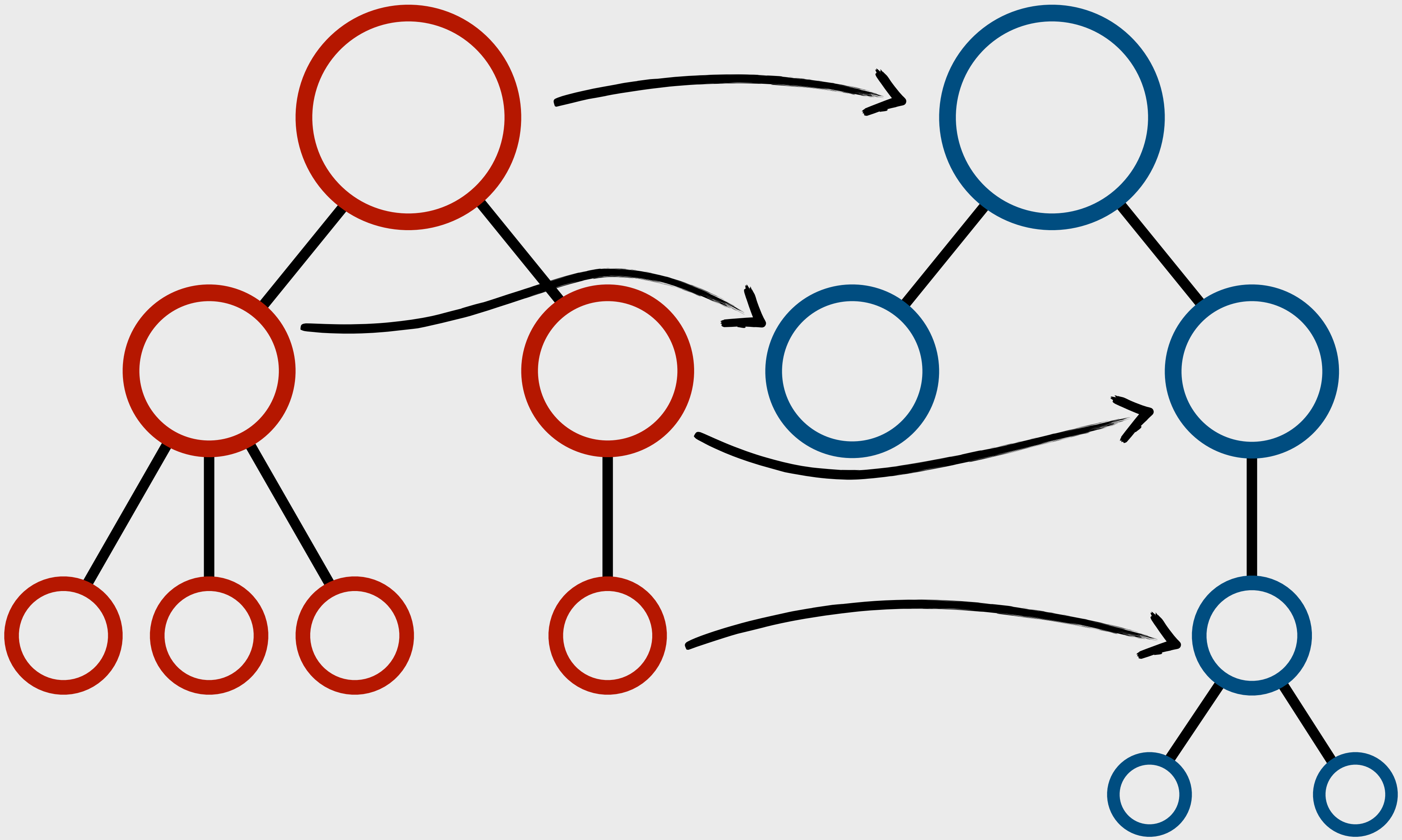
Structure Maps



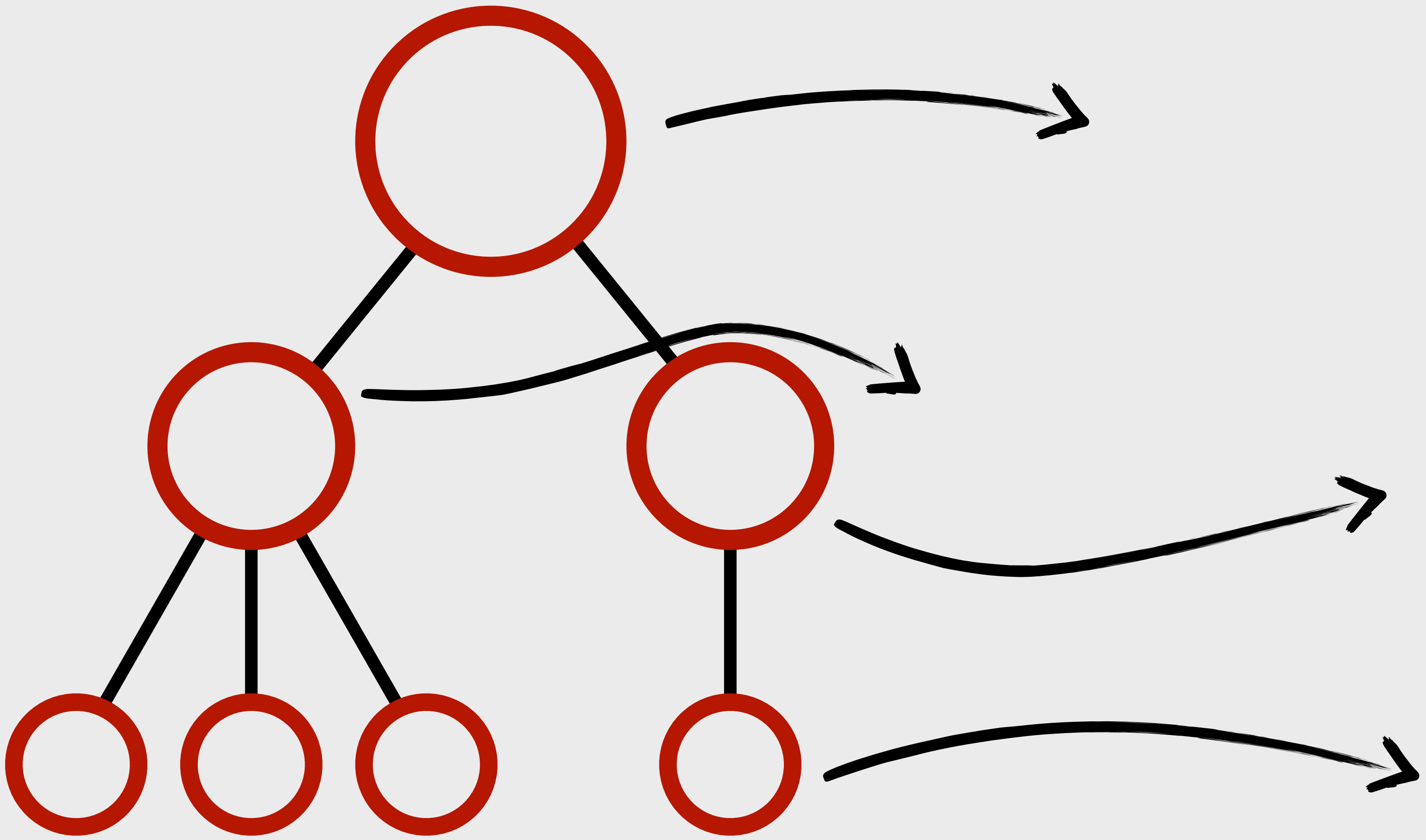
Structure Maps



Structure Maps



Structure Maps



Rules of the Game

Rules

Given statement in first representation

Rules

Given statement in first representation

Set of alternative representations

Rules

Given statement in first representation

Set of alternative representations

No available translations

Rules

Given statement in first representation

Set of alternative representations

No available translations

Small/no applicable datasets

Describing a Representation

Representations

Cohesive set of tokens, types, tactics, patterns, and laws

Representations

Cohesive set of tokens, types, tactics, patterns, and laws

Intuitive boundaries (usually)

Representations

Cohesive set of tokens, types, tactics, patterns, and laws

Intuitive boundaries (usually)

Can be combined in complex ways

Representations

Cohesive set of tokens, types, tactics, patterns, and laws

Intuitive boundaries (usually)

Can be combined in complex ways

Need to describe them!

```
let Algebra = representation

  import tokens from real_numerals;
  import tokens from latin_alphabet;

  mode sentential;
  rigorous true;

  types integer, real, formula, proof;


```

```
tokens |, =, >, < where
  type = integer → integer → bool;
tokens +, -, ×, ÷, ^ where
  type = integer * integer → integer;
token ∑ where
  type = ‘a set → (‘a → integer)
                    → integer;
```

```
pattern binaryOperation where
  holes = {integer: 3,
           integer * integer
             → integer: 1},
  tokens = [=];

laws +associative, +commutative,
     *associative, *commutative, …;
```

```
    tactic rewrite where laws = 1,
                         patterns = 1;
    tactic calc where laws = 0,
                      patterns = 1;
    tactic induction where laws = 2,
                           patterns = 1;

end;
```

Correspondences

Correspondences

Links between representations

Correspondences

Links between representations

What fill the same role?

Correspondences

Links between representations

What fill the same role?

Problem-independent

Correspondences

$$\langle q, r, s \rangle$$

Correspondences

q

r

s

Correspondences

q First representation properties

r

s

Correspondences

q First representation properties

r Second representation properties

s

Correspondences

q First representation properties

r Second representation properties

s Relationship strength

Correspondences

$$\langle\, q \qquad , \qquad r \qquad , \qquad s \,\rangle$$

Correspondences

$$\langle\ q \qquad , \qquad r \qquad , \qquad s\ \rangle$$

type number

Correspondences

$$\langle \; q \qquad , \qquad r \qquad , \qquad s \; \rangle$$

type number

type dot arrangement

Correspondences

$$\langle\ q \qquad , \qquad r \qquad , \qquad s\ \rangle$$

type number

type dot arrangement

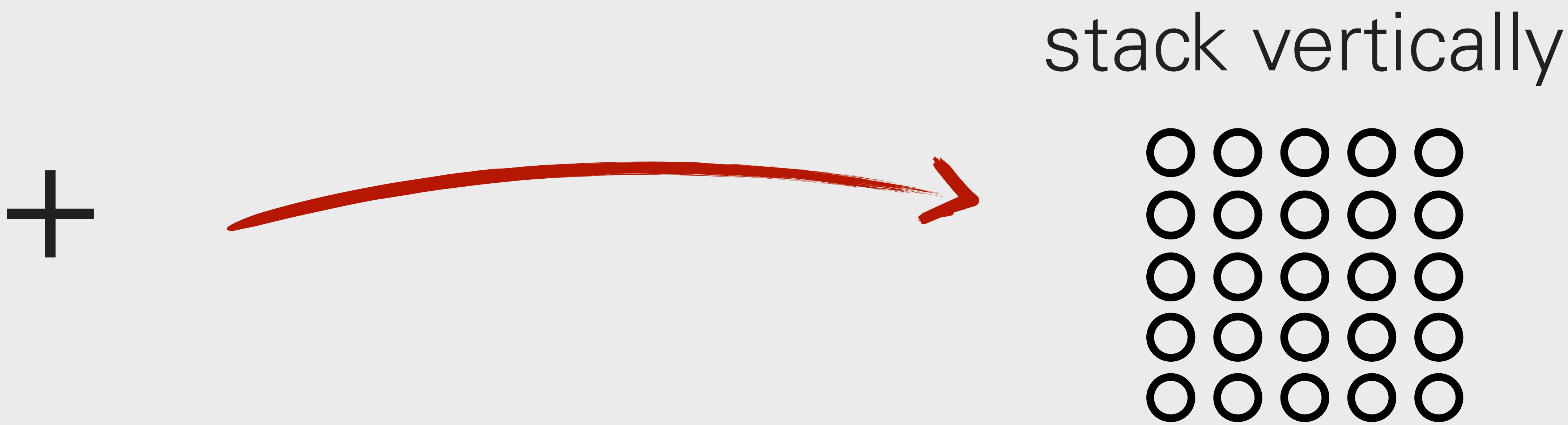
0.9

Property Formulae

Formulae

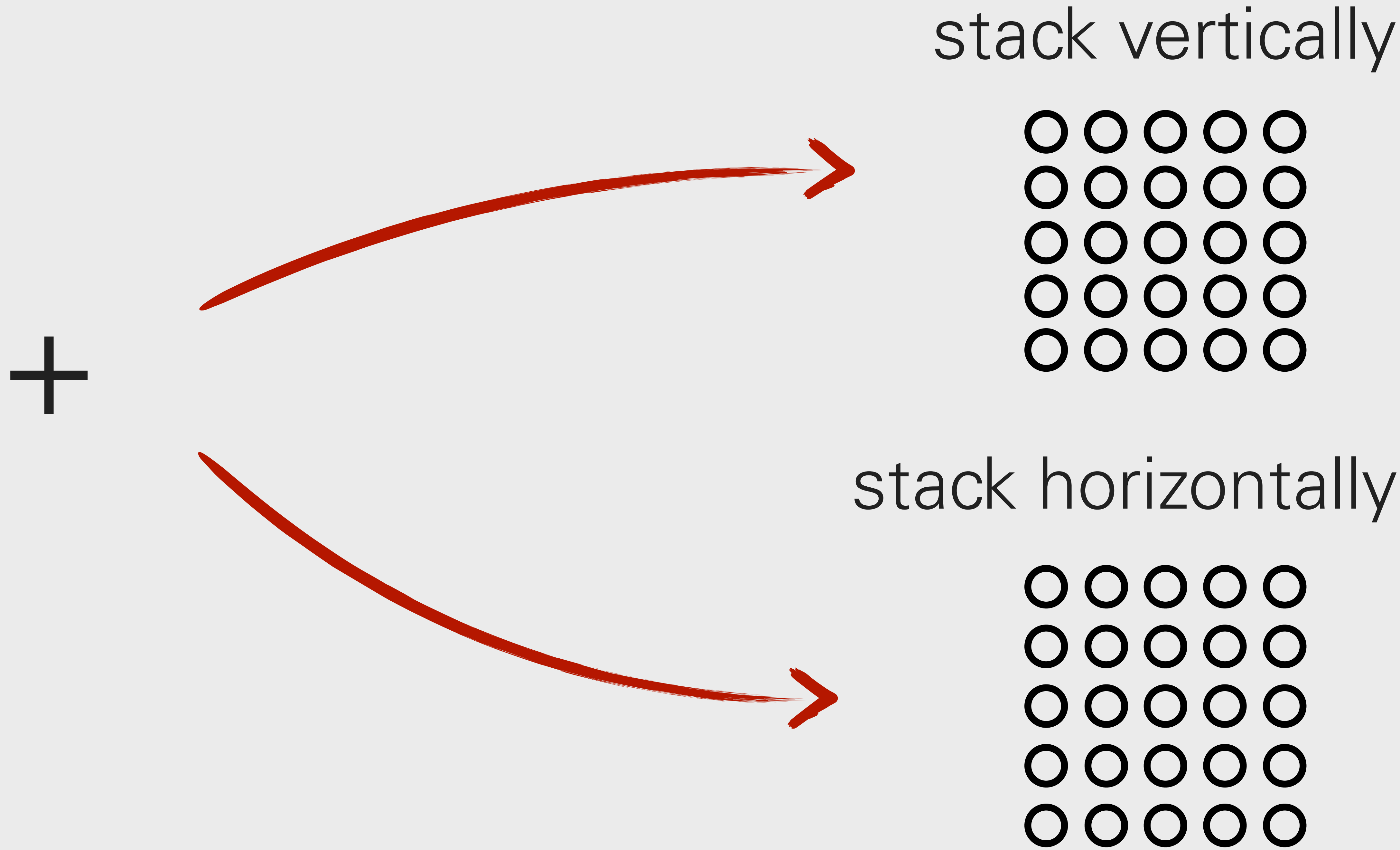
+

Formulae



Formulae

Formulae



Formulae

Alternative related properties

Formulae

Alternative related properties

Requires several properties together

Formulae

Alternative related properties

Requires several properties together

Properties should be absent

Formulae

Alternative related properties OR

Requires several properties together

Properties should be absent

Formulae

Alternative related properties OR

Requires several properties together AND

Properties should be absent

Formulae

Alternative related properties **OR**

Requires several properties together **AND**

Properties should be absent **NOT**

Formulae

Formulae

$\left\langle \text{token} + \right.$,

Formulae

$\langle$ token + ,

tactic stack-horizontal
OR tactic stack-vertical ,

Formulae

$$\Big\langle \text{token} + ,\ \begin{array}{l}\text{tactic stack-horizontal} \\ \text{OR tactic stack-vertical}\end{array},\ 0.9 \Big\rangle$$

Strength

Strength

$\langle$ token + ,

tactic stack-horizontal
OR tactic stack-vertical ,

0.9 $\rangle$

Strength

⟨ token + ,

tactic stack-horizontal
OR tactic stack-vertical ,

0.9 ⟩

Strength

Measure of suitability

Strength

Measure of suitability

Perfect is 1, meaningless is 0

Strength

Measure of suitability

Perfect is 1, meaningless is 0

Any real value in between

Strength

$$s(r \mid q) = \frac{\Pr(r \mid q) - \Pr(r)}{1 - \Pr(r)}$$

Strength

$$s(r \mid q) = \frac{\Pr(r \mid q) - \Pr(r)}{1 - \Pr(r)}$$

Proportion of actual change
to potential change

Strength

Properties have probability

Strength

Properties have probability

Bayesian prior / Frequentist occurrences

Strength

Properties have probability

Bayesian prior / Frequentist occurrences

Knowing one informs another

Strength

$$s(r \mid q) = \frac{\Pr(r \mid q) - \Pr(r)}{1 - \Pr(r)}$$

Strength

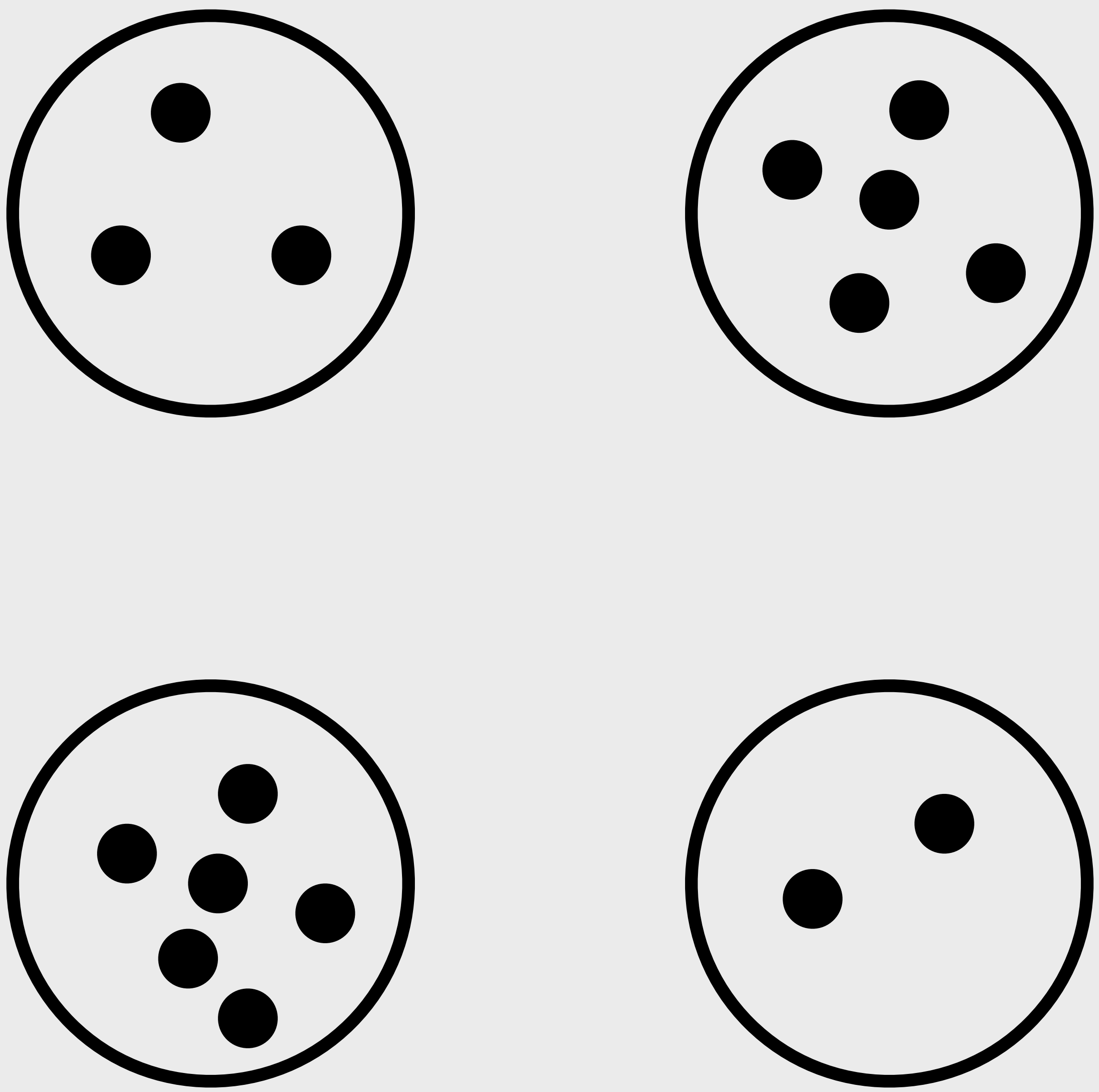
$$s(r \mid q) = \frac{\Pr(r \mid q) - \Pr(r)}{1 - \Pr(r)}$$

Proportion of actual change
to potential change

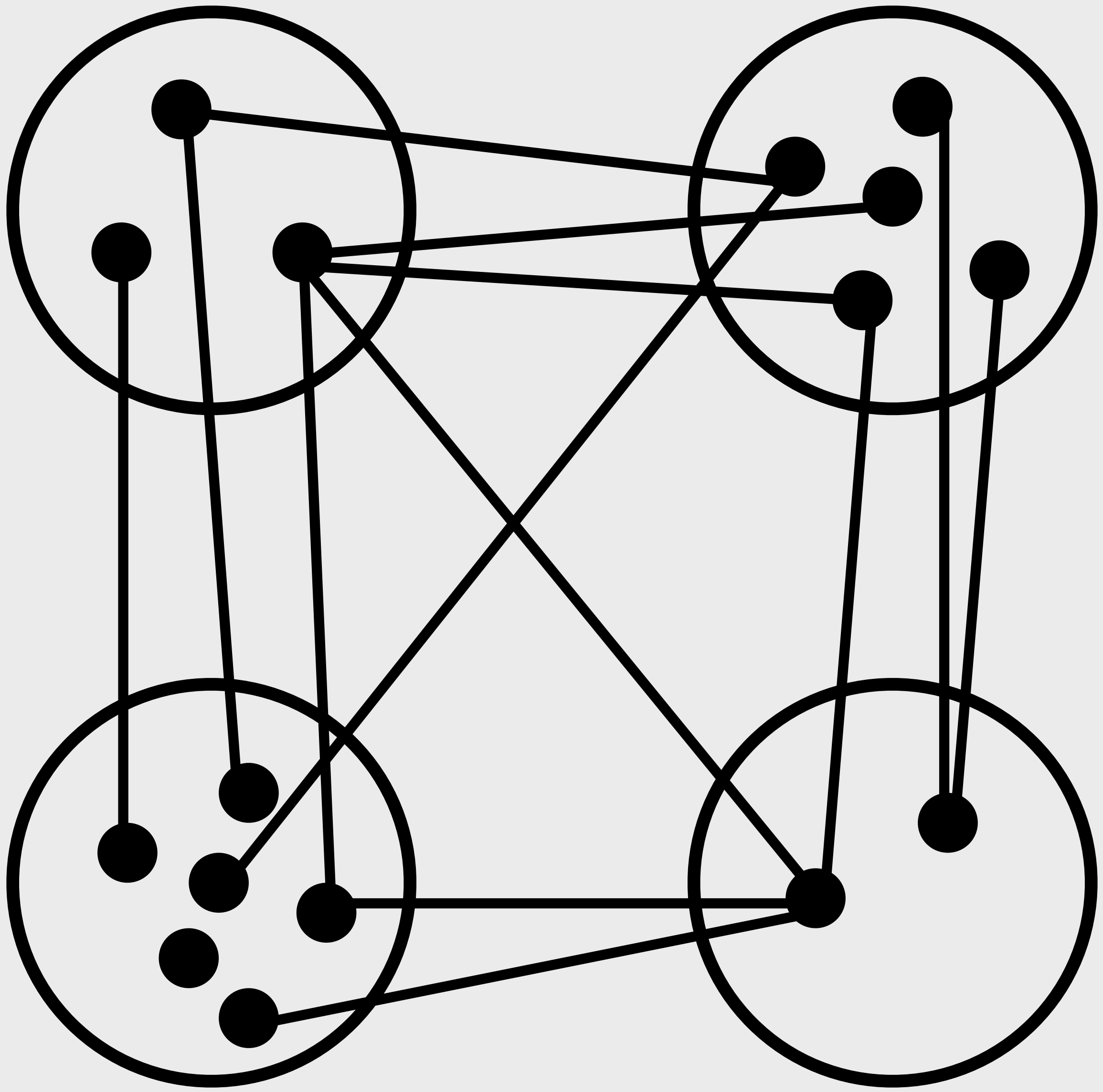
Deriving Correspondences

Derivation

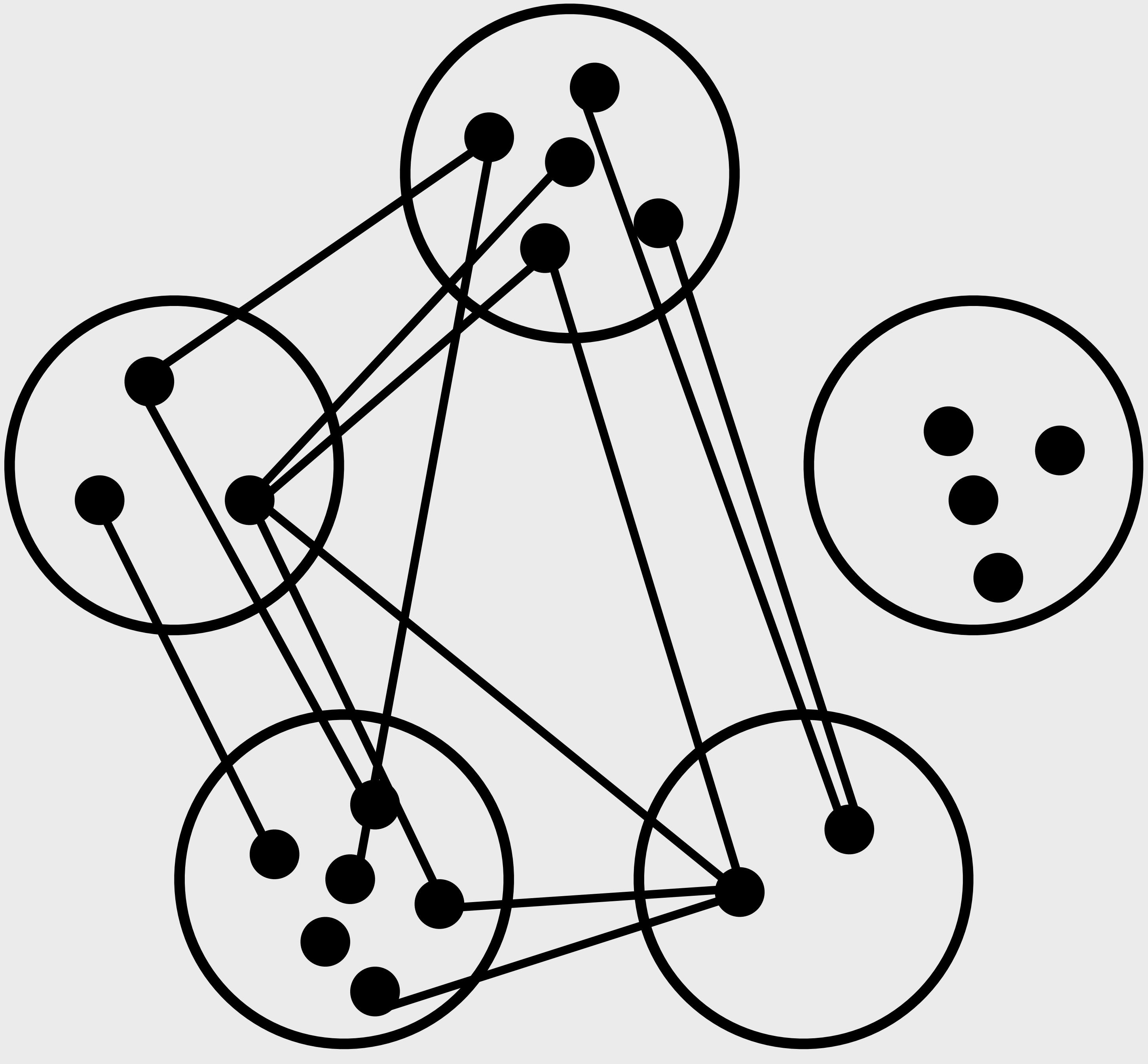
Derivation



Derivation



Derivation



Derivation

Difficult to think of

Derivation

Difficult to think of

Many correspondences

Derivation

Difficult to think of

Many correspondences

More *usually* better

Derivation

Difficult to think of

Many correspondences

More *usually* better

Automate it!

Rules

Rules

$$\overline{\langle a, a, 1 \rangle}$$

If two properties are identical,
they correspond perfectly

Rules

$$\frac{\langle a, b, s \rangle}{\langle b, a, s' \rangle}$$

Correspondences can be reversed

Rules

$$\frac{\langle a, b, s\rangle}{\langle b, a, s'\rangle}$$

Correspondences can be reversed

$$s' = s \cdot \frac{\Pr(a)}{1 - \Pr(a)} \cdot \frac{1 - \Pr(b)}{\Pr(b)}$$

Rules

$$\frac{\langle a, b, s_1 \rangle \quad \langle c, d, s_2 \rangle}{\langle c[b/a], d, s_1 \cdot s_2 \rangle}$$

Correspondences can be chained together

Rules

$$\frac{a\{k = v\} \quad b\{k = v'\} \quad \langle a, b, s\rangle}{\langle v, v', s\rangle}$$

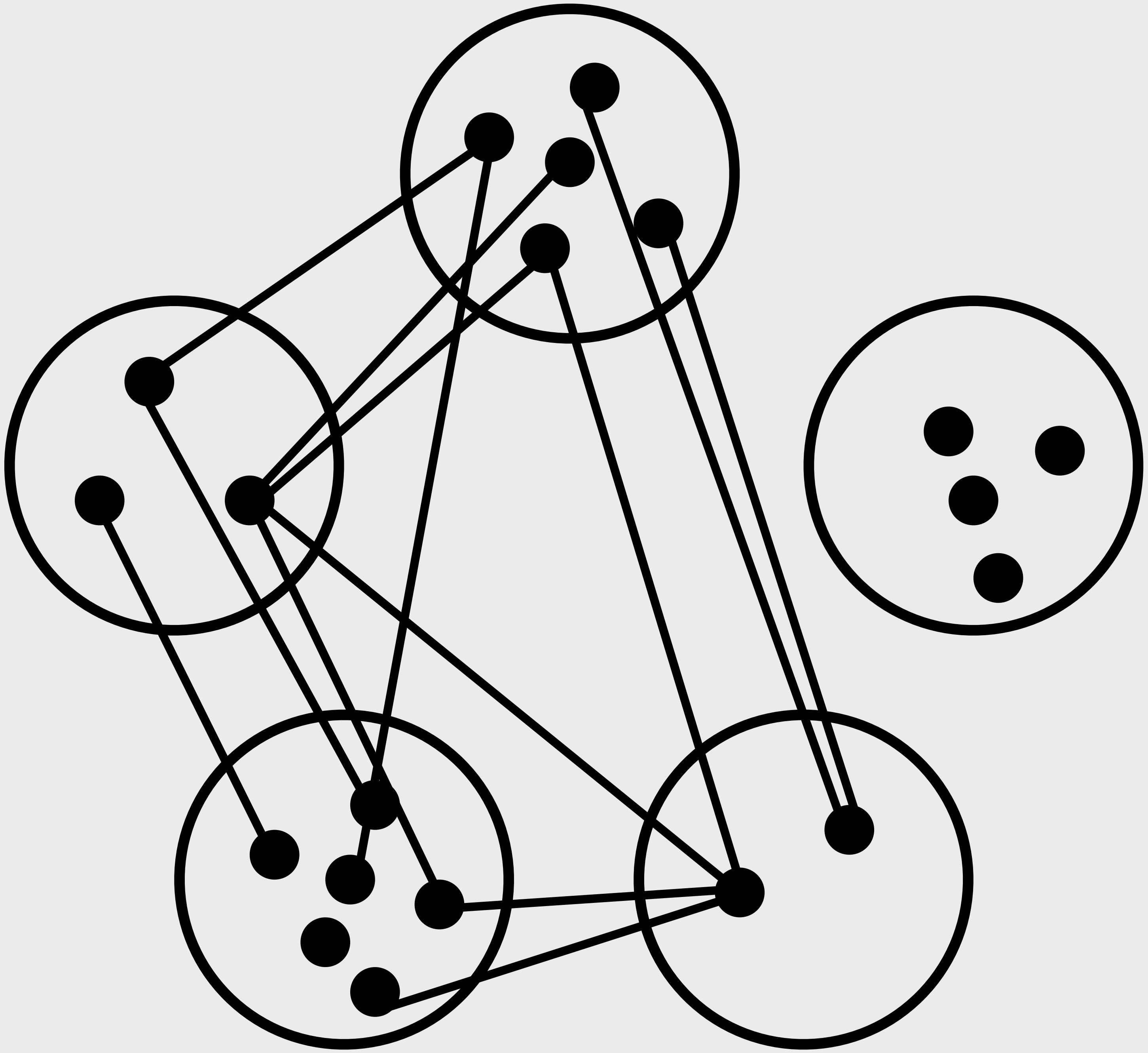
Attributes of corresponding properties
may themselves correspond

Rules

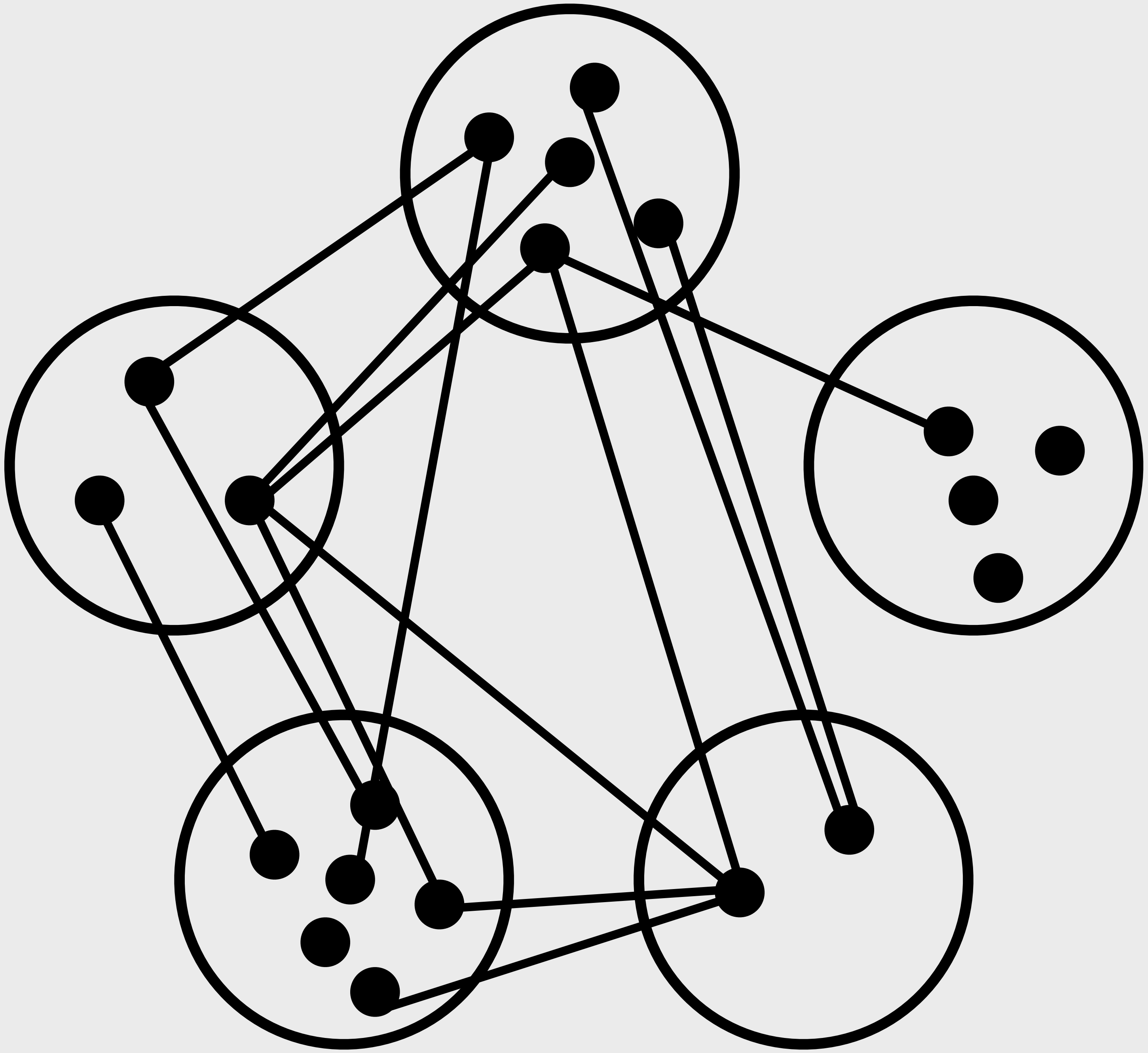
$$\frac{a\{k = v\} \quad b\{k = v'\} \quad \langle v, v', s\rangle}{\langle a, b, s\rangle}$$

Properties with corresponding attributes
may themselves correspond

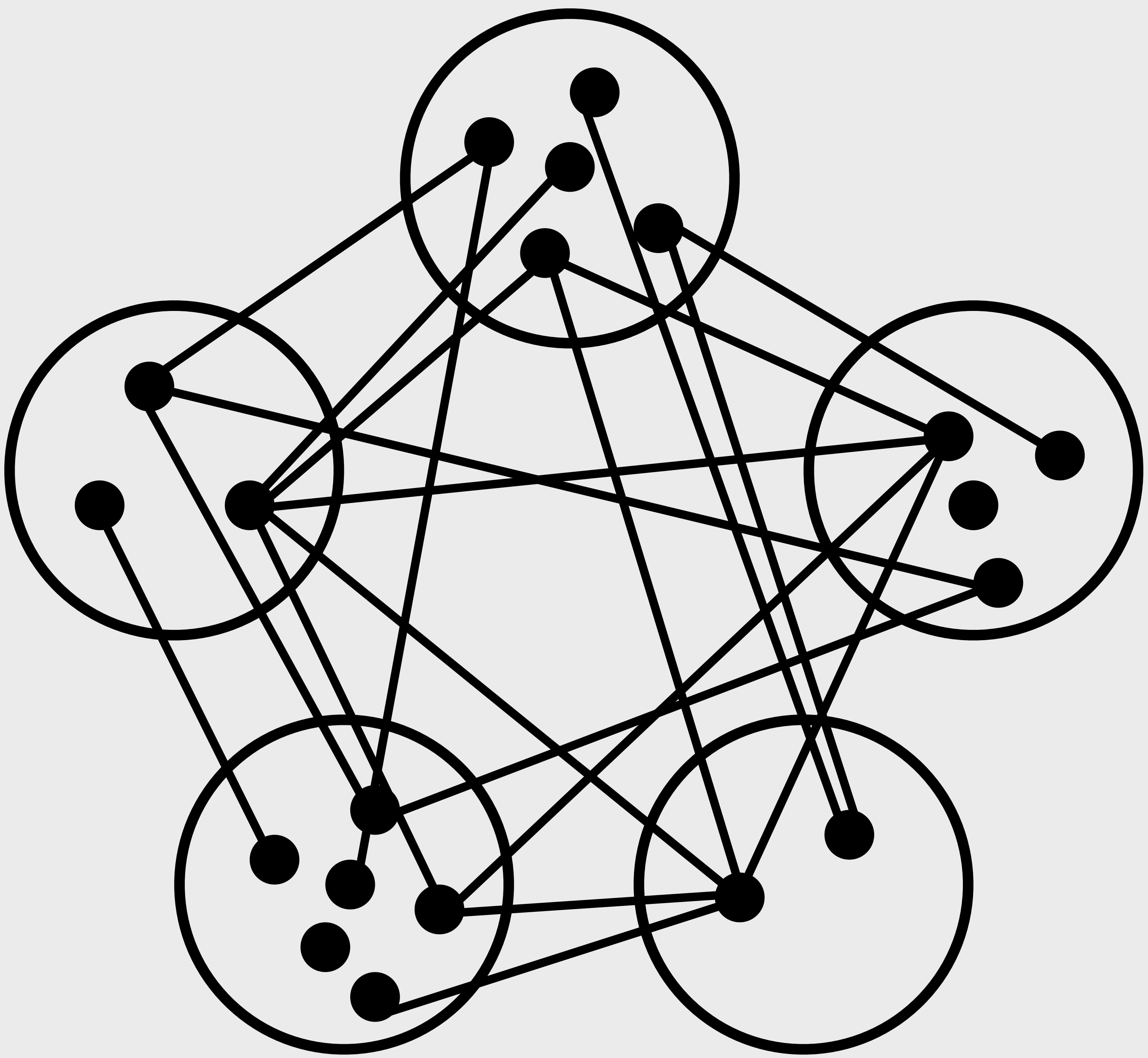
Derivation



Derivation



Derivation



Derivation

$$\left\langle \text{token 1} \quad , \quad \text{token dot} \quad , \quad 1.0 \right\rangle$$

Derivation

$$\left\langle \text{token 1}, \text{token dot}, \text{1.0} \right\rangle$$

Derivation

$$\left\langle \text{token 1} \;,\; \text{token dot} \;,\; 1.0 \right\rangle$$

token 1 : { type = int }

Derivation

$$\langle \text{ token 1 }, \text{ token dot }, \text{ 1.0 } \rangle$$

token 1 : { type = int }

token dot : { type = arrangement }

Derivation

⟨ token 1 , token dot , 1.0 ⟩

token 1 : { type = int }

token dot : { type = arrangement }

Derivation

⟨ token 1 , token dot , 1.0 ⟩

token 1 : { type = int }

token dot : { type = arrangement }

⟨ type int , type arrangement , 1.0 ⟩

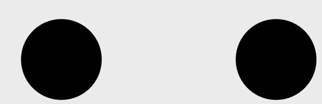
Domination

Domination

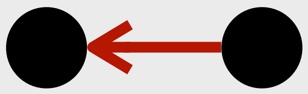
Multiple derivations

Domination

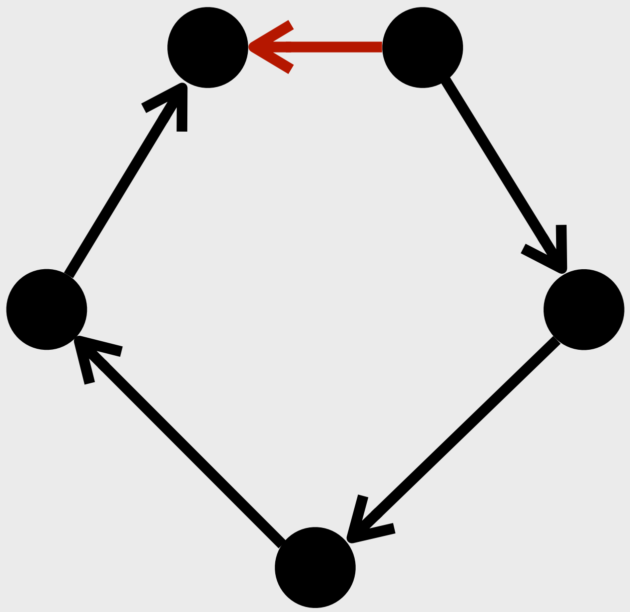
Domination



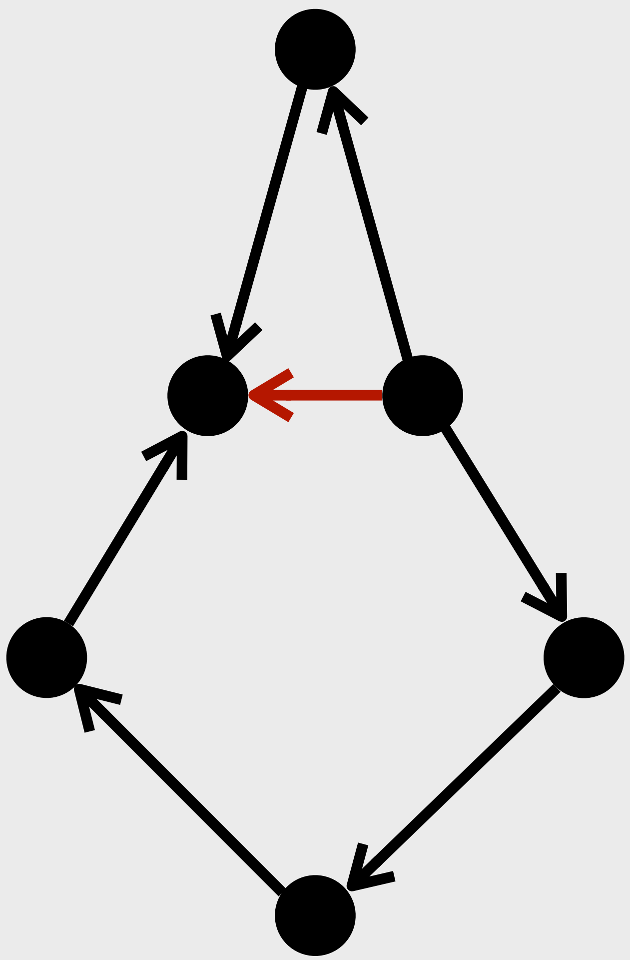
Domination



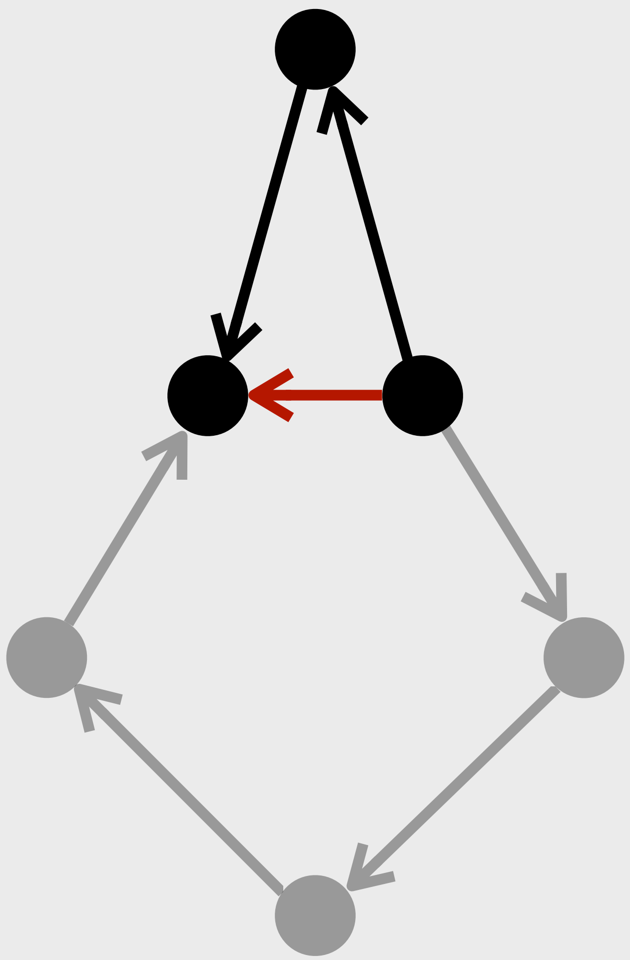
Domination



Domination



Domination



Domination

Multiple derivations

Domination

Multiple derivations

More specific rules, or stronger rules, dominate

Domination

Multiple derivations

More specific rules, or stronger rules, dominate

How to order?

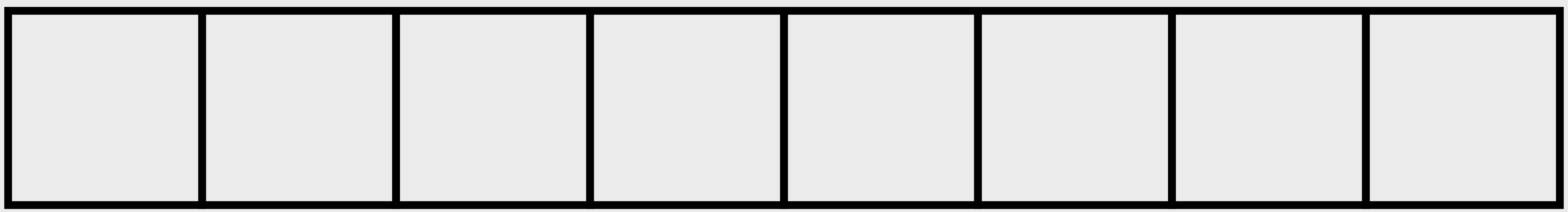
Domination

Multiple derivations

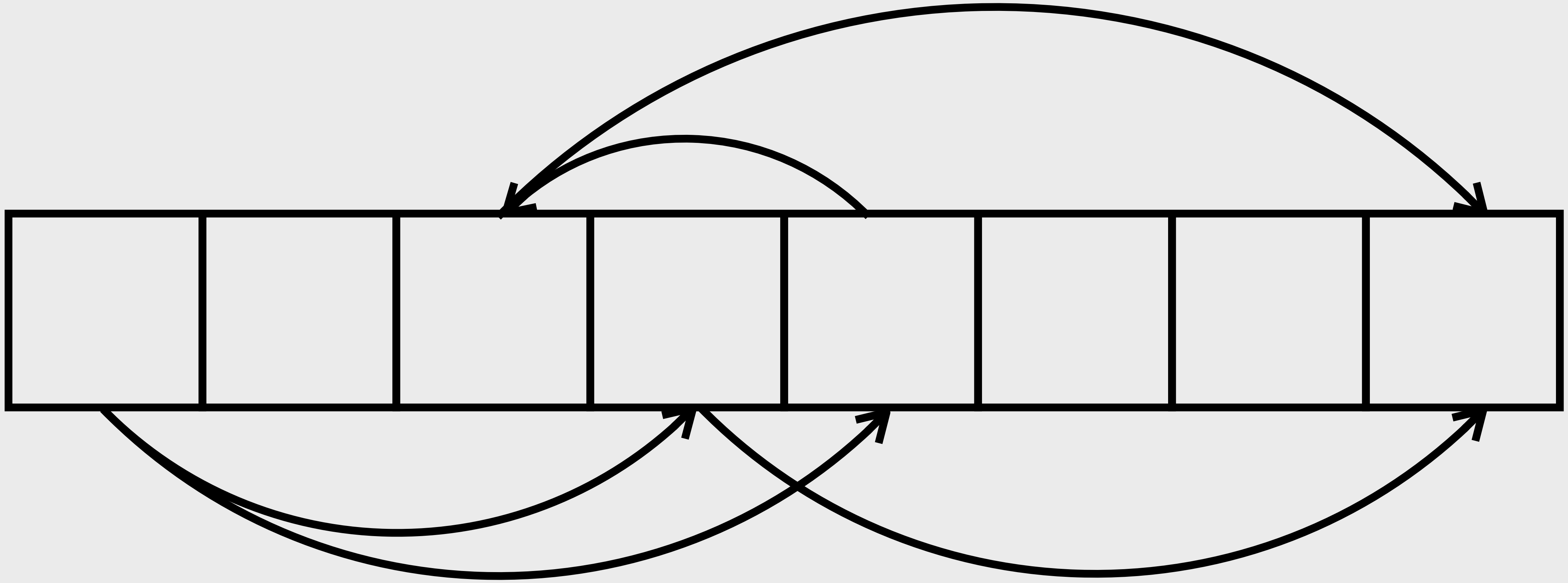
More specific rules, or stronger rules, dominate

How to replace?

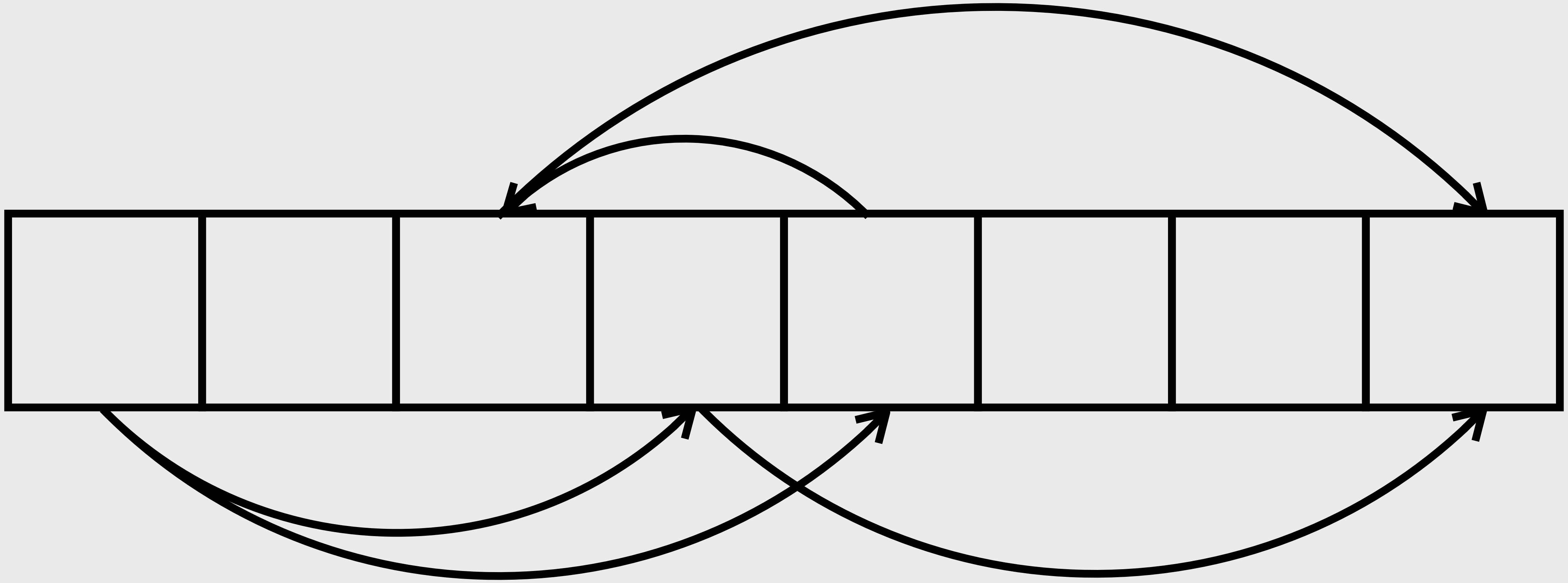
Domination



Domination

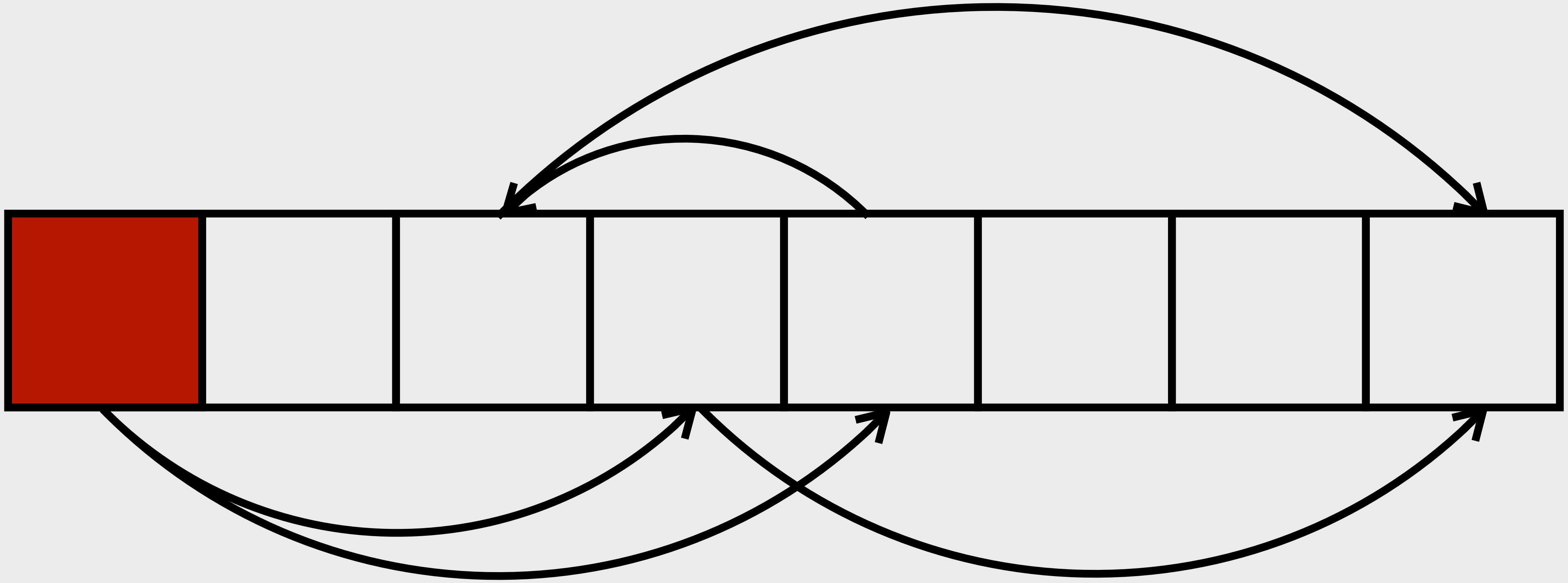


Domination



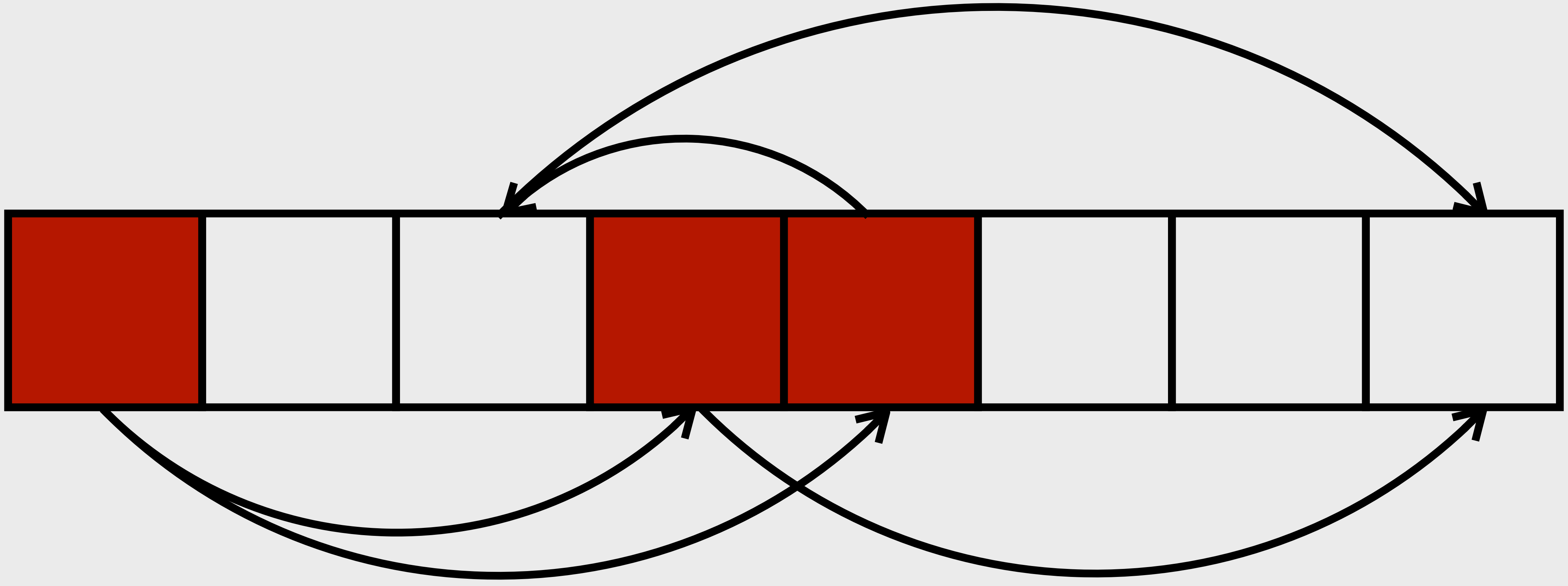
Derivation children

Domination



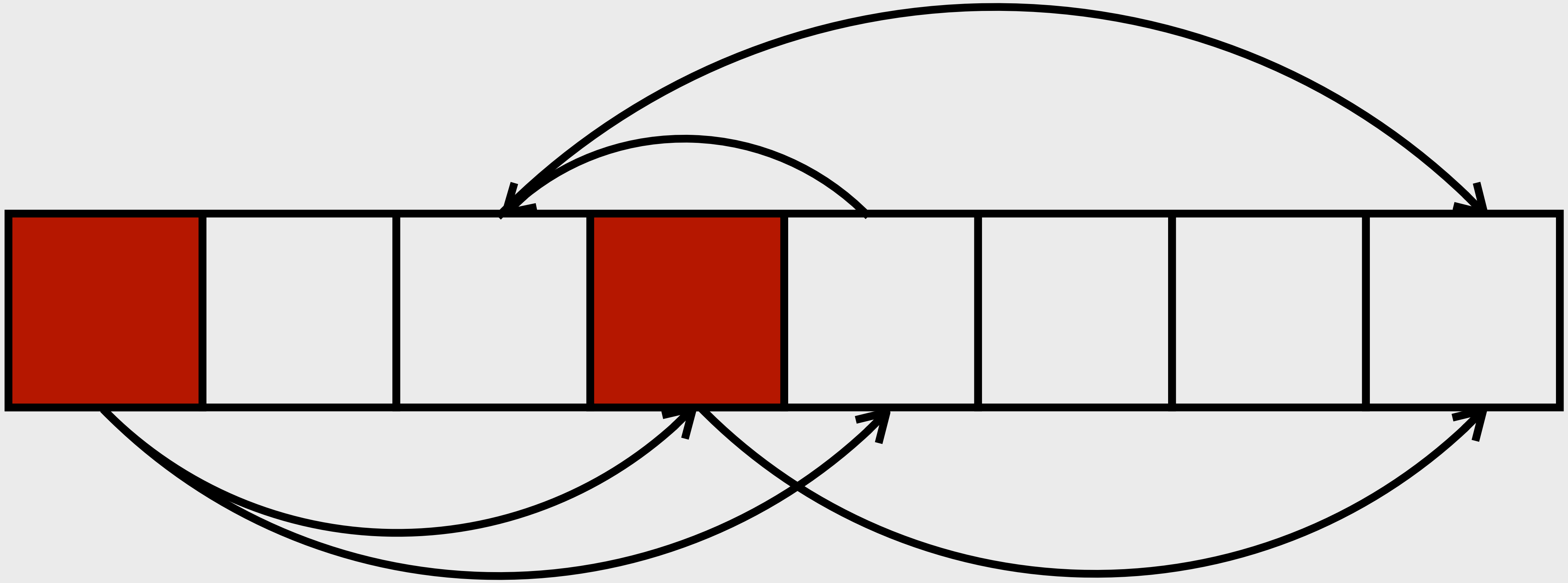
Derivation children

Domination



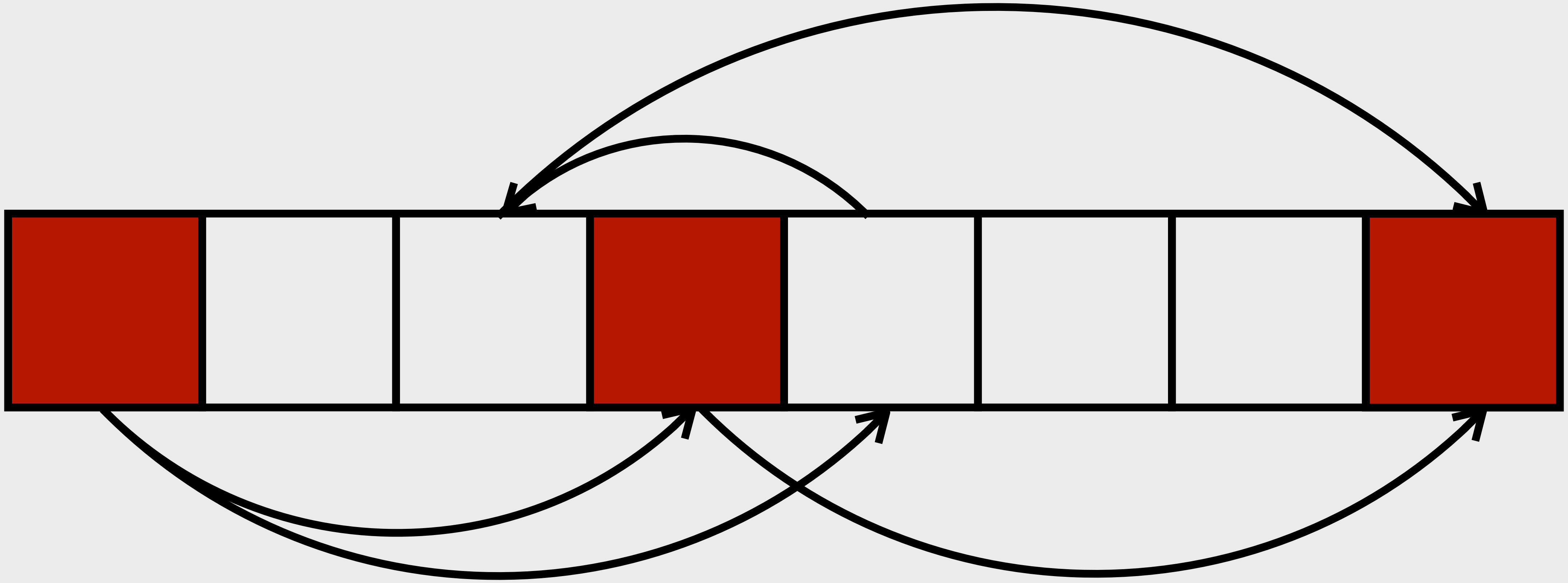
Derivation children

Domination



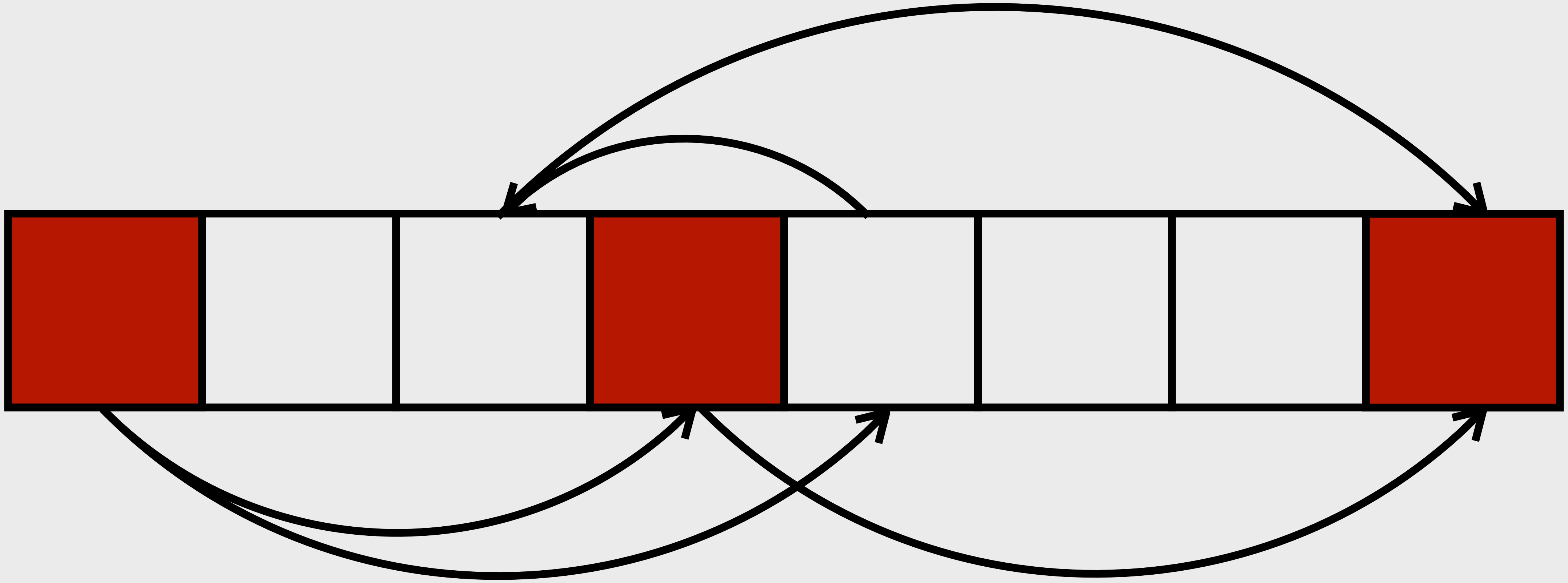
Derivation children

Domination



Derivation children

Domination



Derivation children

Enforce acyclicity (to avoid infinite loops!)

Generalising

Generalising

Recommend other things

Generalising

Recommend other things

Films, books, music

Generalising

Recommend other things

Films, books, music → all together

Generalising

Recommend other things

- *Films, books, music* → all together

Post-hoc rationalisation

Generalising

Set of structures $\mathcal{S}$

Generalising

Set of structures $\mathcal{S}$

Each $S \in \mathcal{S}$ is a tuple $S = (A, \mathcal{R}, \mathrm{Pr})$

Generalising

Set of structures $\mathcal{S}$

Each $S \in \mathcal{S}$ is a tuple $S = (A, \mathcal{R}, \mathrm{Pr})$

Set A contains atoms

Generalising

Set of structures $\mathcal{S}$

Each $S \in \mathcal{S}$ is a tuple $S = (A, \mathcal{R}, \mathrm{Pr})$

Set A contains atoms

Set $\mathcal{R}$ contains predicates on $A^2 \ldots A^n$

Generalising

Set of structures $\mathcal{S}$

Each $S \in \mathcal{S}$ is a tuple $S = (A, \mathcal{R}, \mathrm{Pr})$

Set A contains atoms

Set $\mathcal{R}$ contains predicates on $A^2 \ldots A^n$

Function Pr assigns probabilities to atoms

Generalising

Generalising

$$\frac{}{\langle a, a, 1 \rangle}$$

Generalising

$$\frac{}{\langle a, a, 1 \rangle} \qquad \frac{\langle a, b, s \rangle}{\langle b, a, s' \rangle}$$

Generalising

Reversal formula earlier

$$\frac{}{\langle a, a, 1 \rangle} \qquad \frac{\langle a, b, s \rangle}{\langle b, a, s' \rangle}$$

Generalising

Reversal formula earlier

$$\frac{}{\langle a, a, 1 \rangle} \qquad \frac{\langle a, b, s \rangle}{\langle b, a, s' \rangle} \qquad \frac{\langle a, b, s_1 \rangle \quad \langle c, d, s_2 \rangle}{\langle c[b/a], d, s_1 \cdot s_2 \rangle}$$

Generalising

Reversal formula earlier

$$\frac{}{\langle a, a, 1\rangle} \qquad \frac{\langle a, b, s\rangle}{\langle b, a, s'\rangle} \qquad \frac{\langle a, b, s_1\rangle \quad \langle c, d, s_2\rangle}{\langle c[b/a], d, s_1 \cdot s_2\rangle}$$

$$\frac{P(x_1, \ldots, x_n) \quad P(y_1, \ldots, y_n) \quad \langle x_1, y_1, s_1\rangle \cdots \langle x_k, y_k, s_k\rangle}{\langle x_{k+1}, y_{k+1}, s'\rangle \ \cdots \ \langle x_n, y_n, s'\rangle}$$

Generalising

Reversal formula earlier

$$\frac{}{\langle a, a, 1\rangle} \qquad \frac{\langle a, b, s\rangle}{\langle b, a, s'\rangle} \qquad \frac{\langle a, b, s_1\rangle \quad \langle c, d, s_2\rangle}{\langle c[b/a], d, s_1 \cdot s_2\rangle}$$

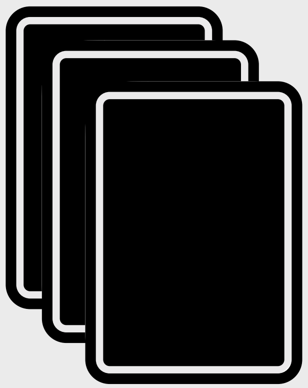
$$\frac{P(x_1, \ldots, x_n) \quad P(y_1, \ldots, y_n) \quad \langle x_1, y_1, s_1\rangle \cdots \langle x_k, y_k, s_k\rangle}{\langle x_{k+1}, y_{k+1}, s'\rangle \ \cdots \ \langle x_n, y_n, s'\rangle}$$

Where $s' = \frac{1}{n-1}\sum_{i=1}^{k} s_i$

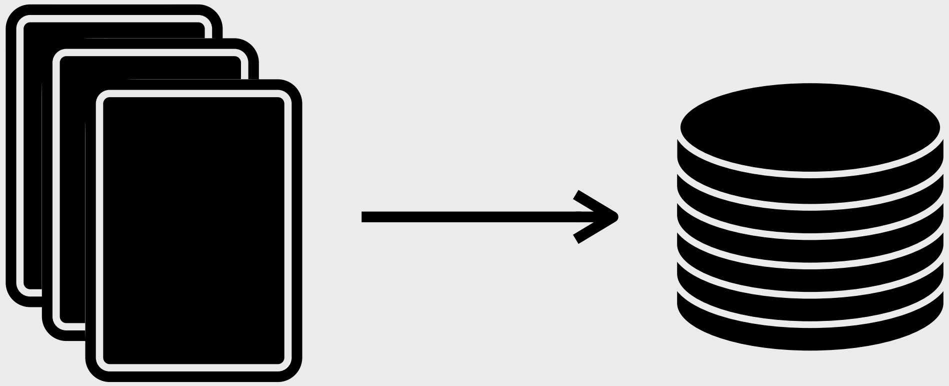
In the rep2rep Framework

rep2rep

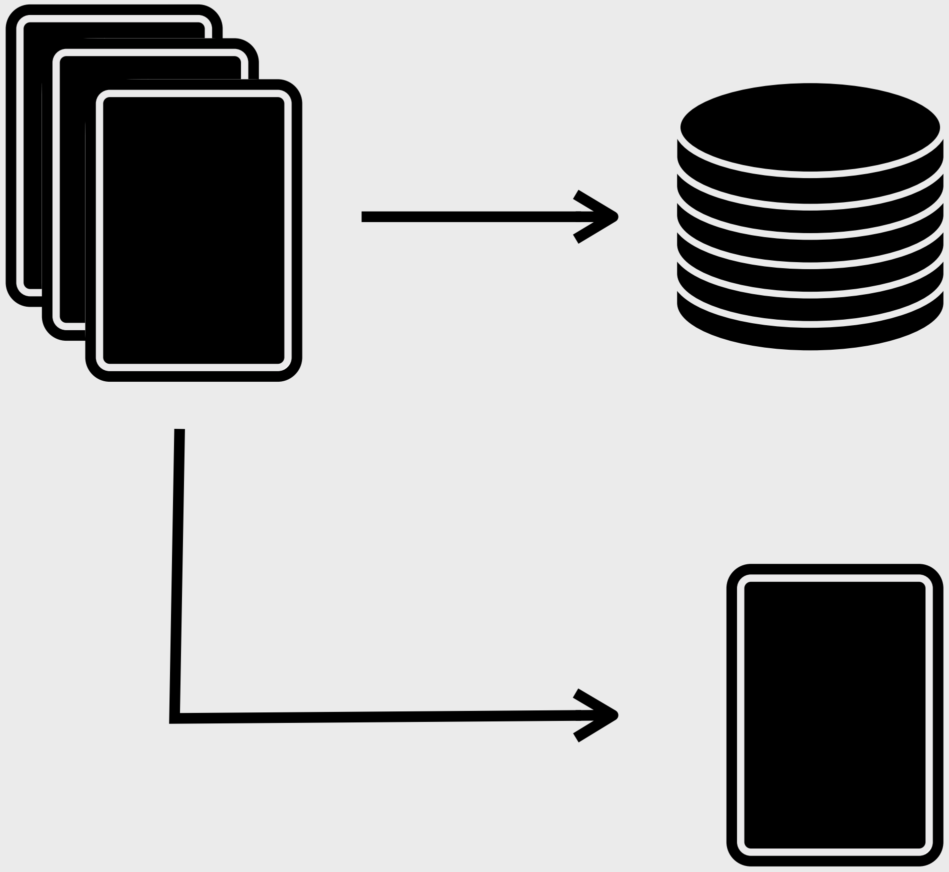
rep2rep



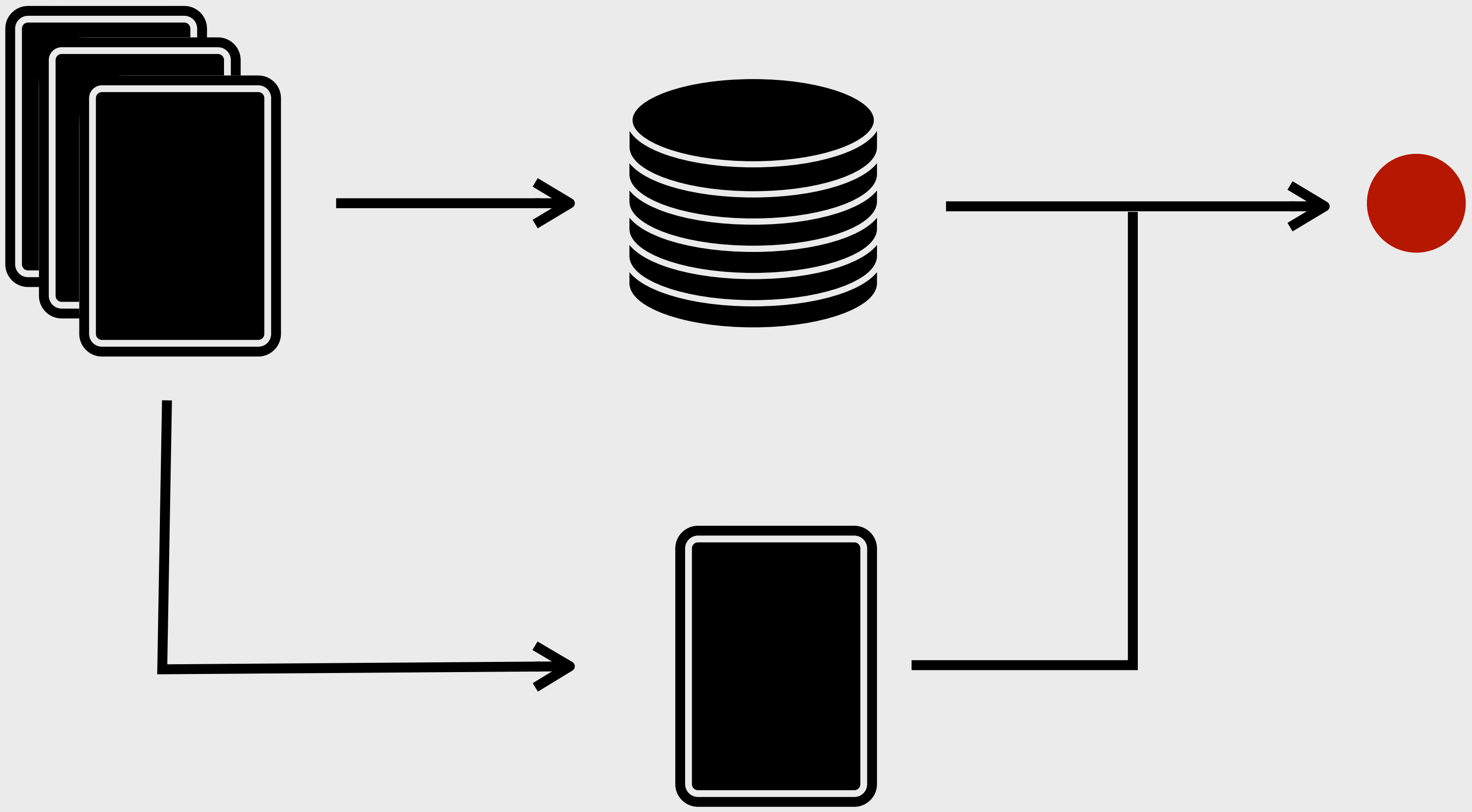
rep2rep



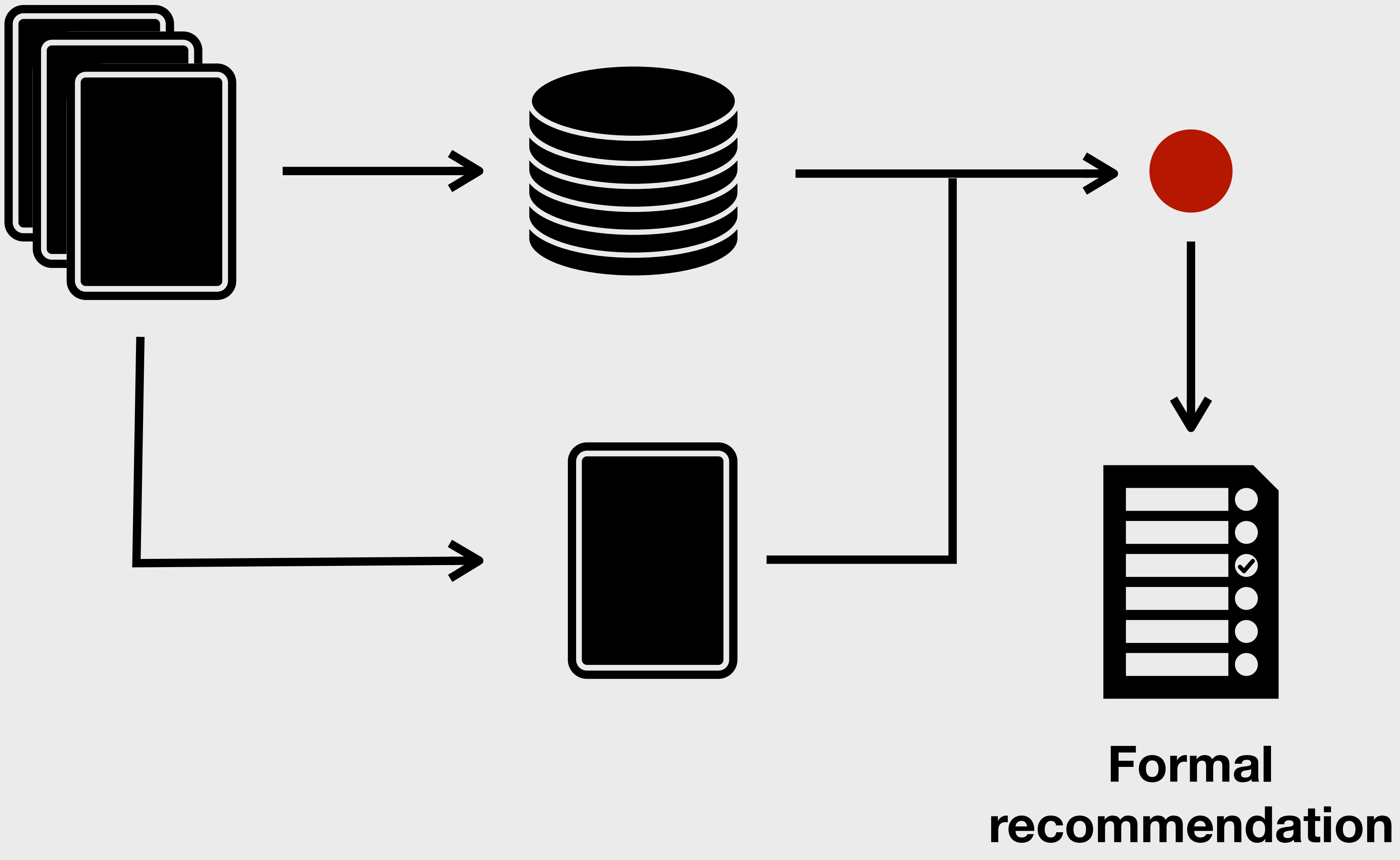
rep2rep



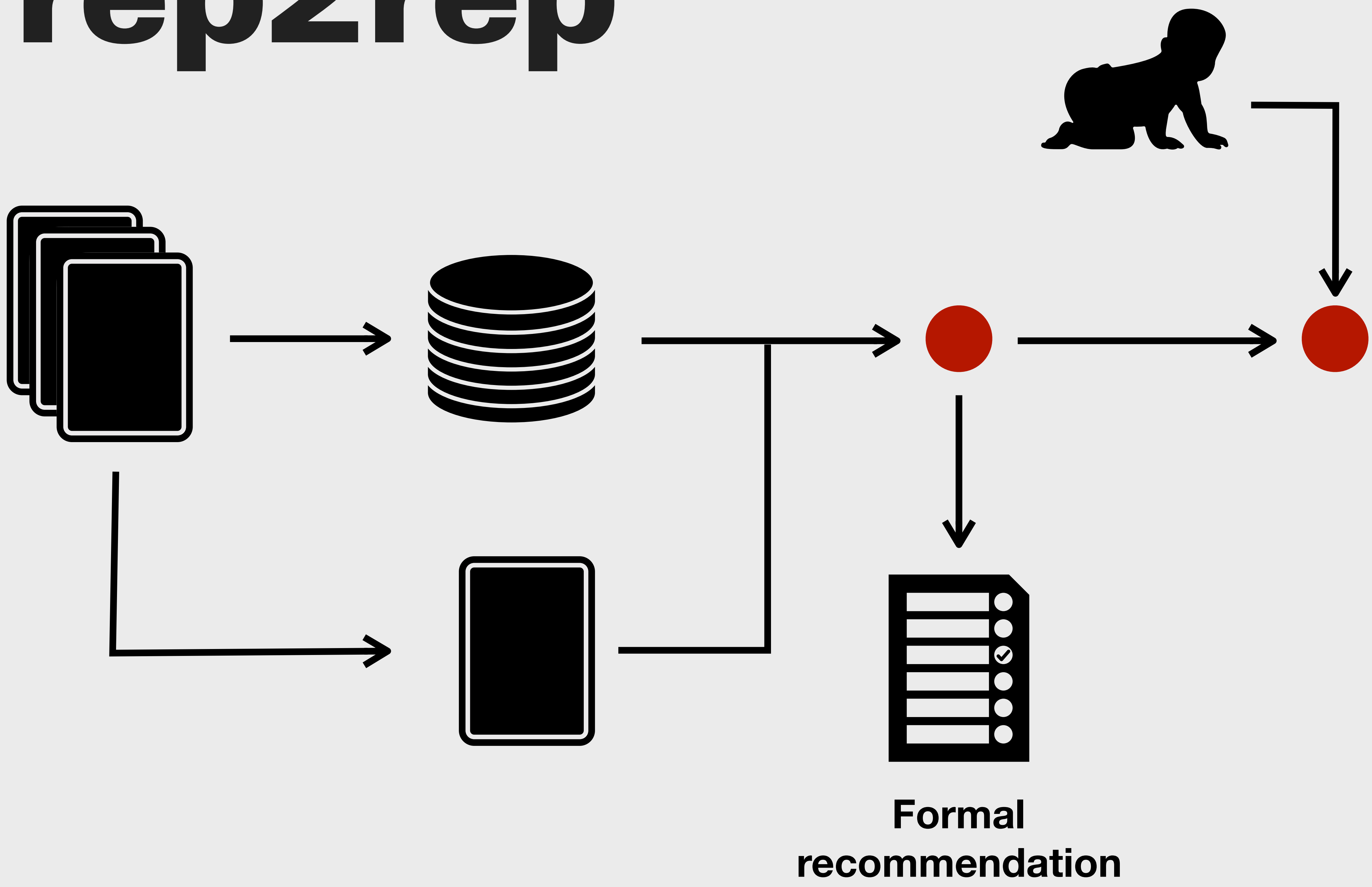
rep2rep

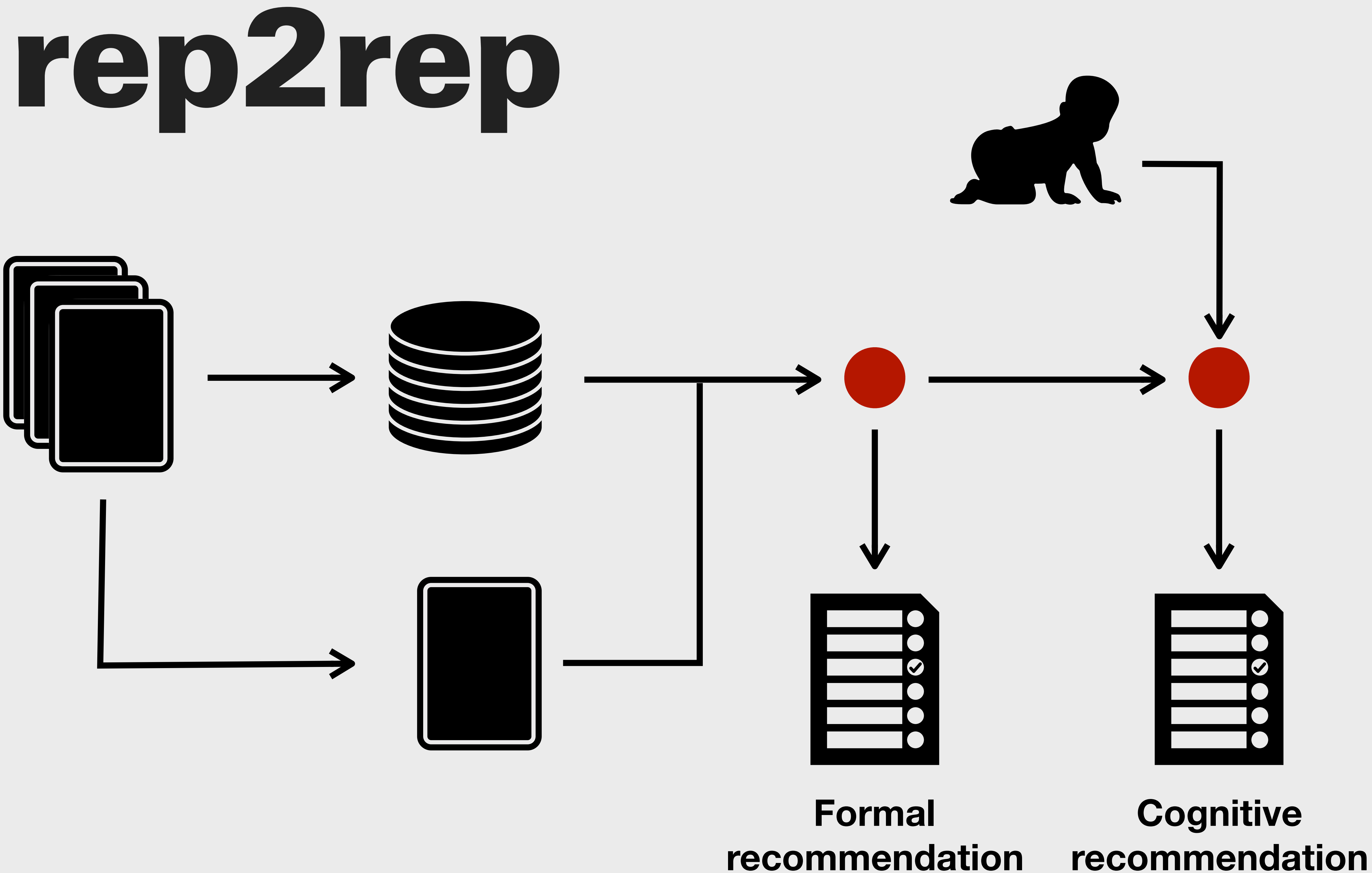


rep2rep

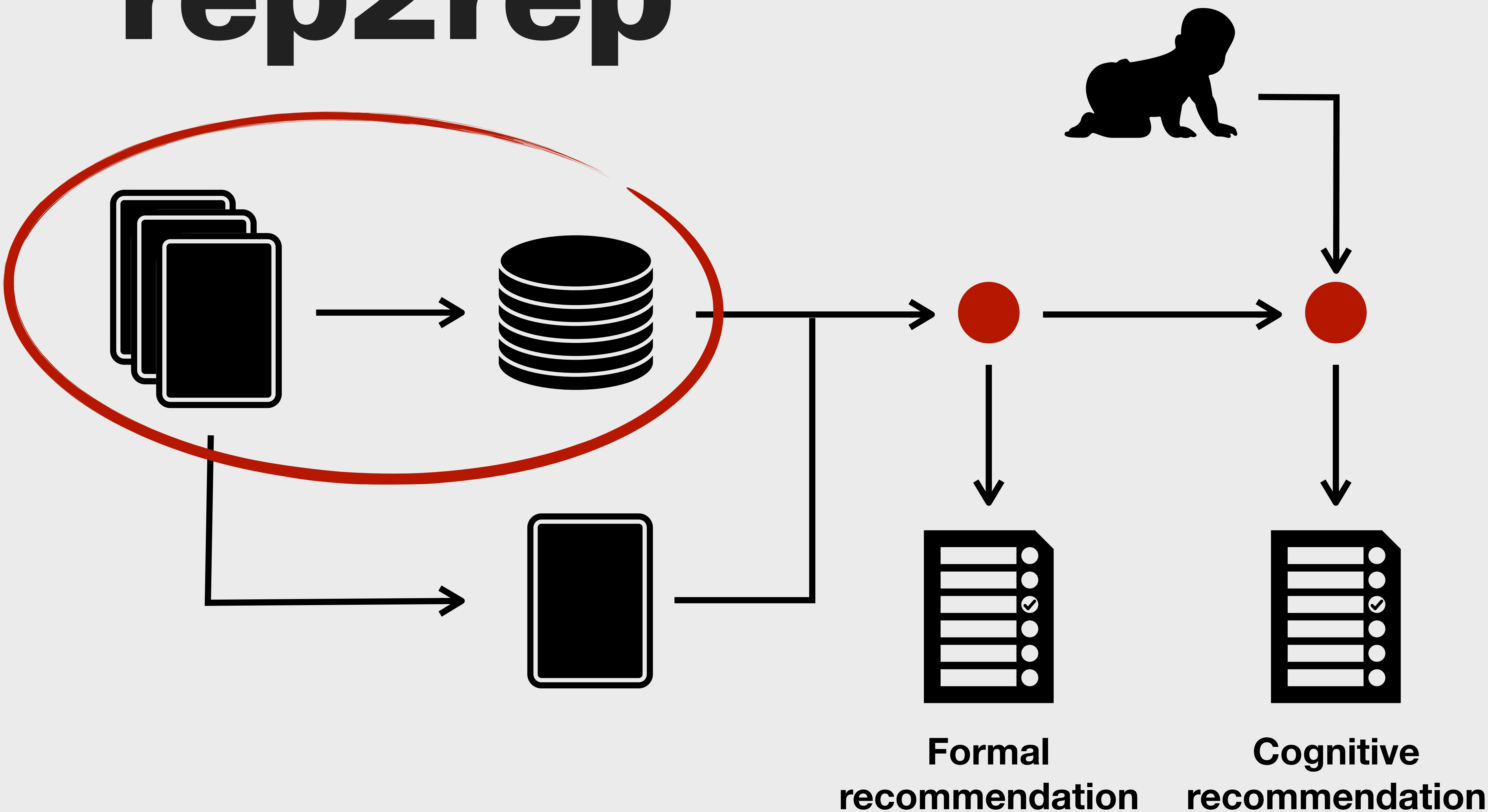


rep2rep




rep2rep
Formal
recommendation
Cognitive
recommendation

rep2rep



Finding an Analogy

Aaron Stockdill

Mateja Jamnik, Daniel Raggi
Peter Cheng, Grecia Garcia Garcia, Holly Sutherland